Reactive Turing Machines

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Abstract

We propose reactive Turing machines, extending classical Turing machines with a process-theoretical notion of interaction. We show that every effective transition system is simulated up to branching bisimilarity by a reactive Turing machine, and that every computable transition system with a bounded branching degree is simulated up to divergence-preserving branching bisimilarity by a reactive Turing machine. We conclude from these results that there exist universal reactive Turing machines, and that the parallel composition of a finite number of (communicating) reactive Turing machines can be simulated by a single reactive Turing machine. We also establish a correspondence between reactive Turing machines and the process theory TCP_τ, proving that a transition system can be simulated, up to branching bisimilarity, by a reactive Turing machine if, and only if, it is definable by a finite TCP_τ-specification.

1 Introduction

The Turing machine [23] is widely accepted as a computational model suitable for exploring the theoretical boundaries of computing. Motivated by the existence of universal Turing machines, many textbooks on the theory of computation (e.g., [22, 18]) present the Turing machine not just as a theoretical model to explain which functions are computable, but as an accurate conceptual model of the computer. For instance, Sipser writes that "[a] Turing machine can do everything a real computer can do." [22] This statement is sometimes referred to as the *strong* Church-Turing thesis, as opposed to the normal Church-Turing thesis according to which every *effectively calculable function* is computable by a Turing machine.

There is, however, a well-known limitation to viewing the Turing machine as a conceptual model of a computer. A Turing machine operates from the assumptions that: (1) all the input it needs for the computation is available on the tape from the very beginning; (2) it performs a terminating computation; and (3) it leaves the output on the tape at the very end. That is, a Turing machine computes a function, and thus it abstracts from two key ingredients of contemporary computing: *interaction* and *non-termination*. Nowadays, most computing systems are so-called *reactive systems* [17], systems that are generally not meant to terminate and typically consist of a number of computing devices that interact with each other and with their environment. A reactive system often unremittingly depends on input, and unremittingly produces output.

Concurrency theory emerged towards the end of the 1970s as the study of reactive systems. Since then, it has contributed significantly to the fundamental understanding of the notion of interaction; we mention three contributions that are particularly relevant for this paper. Firstly, it put forward the notion of labelled transition system —a generalisation of the notion of finite-state automaton from classical automata theory— as the prime mathematical model to represent discrete behaviour. Secondly, it offered the insight that language equivalence — the underlying equivalence in classical automata theory— is too coarse in a setting with interacting automata; instead one should consider automata up to some form of bisimilarity. Thirdly, it yielded many algebraic process calculi facilitating the formal specification and verification of reactive systems.

To date, many undergraduate computer science curricula do provide an introduction to the theory of computation, presenting the Turing machine as a conceptual model of the computer, but then forget to discuss a theory of interaction. In view of the importance of interaction in contemporary computing, we think that this is an unfortunate situation, and that it may be remedied by a better integration of the theories of computation and concurrency. With a better integration of the theories, they can, e.g., be taught efficiently in a single course on automata and processes, as is currently already done at Eindhoven University of Technology. To foster the integration of the theories, we are engaged in a project to reconsider definitions and results of traditional automata theory, as taught in a typical undergraduate course, investigating how to properly extend them with a notion of interaction (see also [3, 8]).

In this paper we propose and discuss a notion of *reactive* Turing machine (RTM), extending the classical notion of Turing machine with interaction in the style of concurrency theory. The extension consists of a facility to declare every transition to be either *observable*, by labelling it with an action symbol, or unobservable, by labelling it with τ. Typically, a transition labelled with an action symbol models an interaction of the RTM with its environment (or some other RTM), while a transition labelled with τ refers to an internal computation step. Thus, a conventional Turing machine can be regarded as a special kind of RTM in which all transitions are declared unobservable by labelling them with τ. The behaviour of RTMs is defined in terms of labelled transition systems, so that they can be considered modulo any suitable behavioural equivalence defined on labelled transition systems. In this paper we shall mainly use *(divergence-preserving) branching bisimilarity* [15], which is the finest behavioural equivalence in van Glabbeek's spectrum (see [13]).

We establish in Sect. 3 that every computable transition system with a bounded branching degree can be simulated by an RTM. This result has some interesting consequences. It allows us to conclude that the behaviour of a parallel composition of RTMs can be simulated on a single RTM. It also allows us to conclude the existence of *universal* RTMs, which can input the code of some arbitrary RTM and then simulate it.

In Sect. 4, we consider the correspondence between RTMs and the process theory TCP_{τ}. We establish that a transition system can be simulated by an RTM if, and only if, it is definable, again up to divergence-preserving branching bisimilarity, by a finite recursive TCP_{τ}-specification [1]. Recursive specifications are often considered to be the process-theoretic counterparts of grammars in the theory of formal languages. So the result in Sect. 4 may be considered as the process-theoretic counterpart of the correspondence between Turing machines and unrestricted grammars. Furthermore, the finite recursive TCP_{τ}-specification actually consists of a specification of the finite-state control of the RTM that interacts with a specification modelling a tape. Thus, as an interesting corollary, we obtain a specification that makes the conceptual interaction within a Turing machine between its finite-state control and its tape memory explicit; a similar result was also obtained for pushdown automata in [5].

Several extensions of Turing machines with some form of interaction have been proposed in the literature, already by Turing in [24], and more recently in [12, 16, 26]. The goal in these works is mainly to investigate to what extent interaction may have a beneficial effect on the power of sequential computation. Interaction is, e.g., added by allowing an algorithm to query its environment, or by assuming that the environment periodically writes a write-only input tape and reads a read-only output tape of a Turing machine. Thereby, the focus remains on the computational aspect, and interaction is not treated as a first-class citizen, whereas our goal is to achieve an integration of automata theory and concurrency theory that treats computation and reactivity on an equal footing.

2 Reactive Turing Machines

We fix a finite set $\mathcal{A}$ of *action symbols* that we shall use to denote the observable events of a system. An unobservable event will be denoted with τ, assuming that $\tau \notin \mathcal{A}$; we shall henceforth denote the set $\mathcal{A} \cup \{\tau\}$ by $\mathcal{A}_\tau$. We also fix a finite set $\mathcal{D}$ of *data symbols*. We add to $\mathcal{D}$ a special symbol $\Box$ to denote a blank tape cell, assuming that $\Box \notin \mathcal{D}$; we denote the set $\mathcal{D} \cup \{\Box\}$ of *tape symbols* by $\mathcal{D}_\Box$.

Definition 2.1. *A* reactive Turing machine *(RTM)* $\mathcal{M}$ *is a quadruple* $(\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ *consisting of a finite set of* states $\mathcal{S}$*, a distinguished* initial state $\uparrow \in \mathcal{S}$*, a subset of* final states $\downarrow \subseteq \mathcal{S}$*, and a* $(\mathcal{D}_\Box \times \mathcal{A}_\tau \times \mathcal{D}_\Box \times \{L, R\})$*-labelled transition relation*

$$\rightarrow \subseteq \mathcal{S} \times \mathcal{D}_\Box \times \mathcal{A}_\tau \times \mathcal{D}_\Box \times \{L, R\} \times \mathcal{S} \ .$$

An RTM is deterministic *if* $(s, d, a, e_1, M_1, t_1) \in \rightarrow$ *and* $(s, d, a, e_2, M_2, t_2) \in \rightarrow$ *implies that* $e_1 = e_2$*,* $t_1 = t_2$ *and* $M_1 = M_2$ *for all* $s, t_1, t_2 \in \mathcal{S}$*,* $d, e_1, e_2 \in \mathcal{D}_\Box$*,* $a \in \mathcal{A}_\tau$*, and* $M_1, M_2 \in \{L, R\}$*.*

If $(s, d, a, e, M, t) \in \rightarrow$, we write $s \xrightarrow{a[d/e]M} t$. The intuitive meaning of such a transition is that whenever $\mathcal{M}$ is in state s and d is the symbol currently read by the tape head, then it may execute the action a, write symbol e on the tape (replacing d), move the read/write head one position to the left or one position to the right on the tape (depending on whether $M = L$ or $M = R$), and then continue in state t. RTMs extend conventional Turing machines by associating with every transition an element $a \in \mathcal{A}_\tau$. The symbols in $\mathcal{A}$ are thought of as denoting observable activities; a transition labelled with an action symbol in $\mathcal{A}$ will semantically be treated as observable. Observable transitions are used to model interactions of an RTM with its environment or some other RTM, as will be explained more in detail below when we introduce a notion of parallel composition for RTMs. The symbol τ is used to declare that a transition is unobservable. We consider a conventional Turing machine as an RTM in which all transitions are declared unobservable.

Example 2.2. *Assume that* $\mathcal{A} = \{c!d, c?d \mid c \in \{i, o\} \ \& \ d \in \mathcal{D}_\Box\}$*. Intuitively,* i *and* o *are the input/output communication channels by which the RTM can interact with its environment. The action symbol* $c!d$ *(*$c \in \{i, o\}$*) then denotes the event that a datum* d *is sent by the RTM along channel* c*, and the action symbol* $c?d$ *(*$c \in \{i, o\}$*) denotes the event that a datum* d *is received by the RTM along channel* c*.*

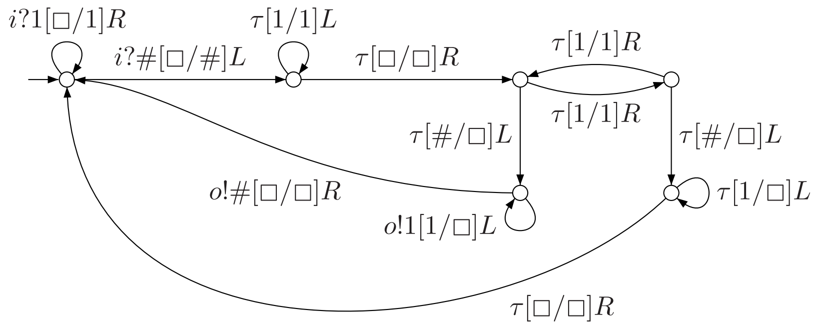

Figure 1: An example of a reactive Turing machine.

The state-transition diagram in Figure 1 concisely specifies an RTM that first inputs a string, consisting of an arbitrary number of 1*s followed by the symbol* #*,* *stores the string on the tape, and returns to the beginning of the string. Then, it performs a computation to determine if the number of* 1*s is odd or even. In the first case, it simply removes the string from the tape and returns to the initial state. In the second case, it outputs the entire string, removes it from the tape, and returns to the initial state.*

To formalise our intuitive understanding of the operational behaviour of RTMs we shall below associate with every RTM a transition system.

Definition 2.3. *An* $\mathcal{A}_\tau$-labelled ransition system T *is a quadruple* $(\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ *consisting of a set of* states $\mathcal{S}$*, an* initial *state* $\uparrow \in \mathcal{S}$*, a subset* $\downarrow \subseteq \mathcal{S}$ *of* final *states, and an* $\mathcal{A}_\tau$*-labelled* transition relation $\rightarrow \subseteq \mathcal{S} \times \mathcal{A}_\tau \times \mathcal{S}$*. If* $(s, a, t) \in \rightarrow$*, we write* $s \xrightarrow{a} t$*. If* s *is a final state, i.e.,* $s \in \downarrow$*, we write* $s\downarrow$*.*

With every RTM $\mathcal{M}$ we are going to associate a transition system $\mathcal{T}(\mathcal{M})$. The states of $\mathcal{T}(\mathcal{M})$ are the configurations of the RTM, consisting of a state of the RTM, its tape contents, and the position of the read/write head on the tape. We represent the tape contents by an element of $(\mathcal{D}_\Box)^*$, replacing precisely one occurrence of a tape symbol d by a *marked* symbol $\check{d}$, indicating that the read/write head is on this symbol. We denote by $\check{\mathcal{D}_\Box} = \{\check{d} \mid d \in \mathcal{D}_\Box\}$ the set of *marked* tape symbols; a *tape instance* is a sequence $\delta \in (\mathcal{D}_\Box \cup \check{\mathcal{D}_\Box})^*$ such that δ contains exactly one element of $\check{\mathcal{D}_\Box}$. Note that we do not use δ exclusively for tape instances; we also use δ for sequences over $\mathcal{D}$. A tape instance thus is a finite sequence of symbols that represents the contents of a two-way infinite tape. Henceforth, we shall not distinguish between tape instances that are equal modulo the addition or removal of extra occurrences of the symbol $\Box$ at the left or right extremes of the sequence. That is, we shall not distinguish tape instances δ_1 and δ_2 if $\Box^\omega\delta_1\Box^\omega = \Box^\omega\delta_2\Box^\omega$.

Definition 2.4. *A* configuration *of an RTM* $\mathcal{M} = (\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ *is a pair* (s, δ) *consisting of a state* $s \in \mathcal{S}$*, and a tape instance* δ*.*

Our transition system semantics defines an $\mathcal{A}_\tau$-labelled transition relation on configurations such that an RTM-transition $s \xrightarrow{a[d/e]M} t$ corresponds with a-labelled transition from configurations consisting of the RTM-state s and a tape instance in which some occurrence of d is marked. The transition leads to a configuration consisting of t and a tape instance in which the marked symbol d is replaced by e, and either the symbol to the left or to right of this occurrence of e is replaced by its marked version, according to whether $M = L$ or $M = R$. If e happens to be the first symbol and $M = L$, or the last symbol and $M = R$, then an additional blank symbol is appended at the left or right end of the tape instance, respectively, to model the movement of the head.

It is convenient to introduce some notation to be able to concisely denote the new placement of the tape head marker. Let δ be an element of $\mathcal{D}_\Box^*$. Then by $\delta^<$ we denote the element of $(\mathcal{D}_\Box \cup \check{\mathcal{D}_\Box})^*$ obtained by placing the tape head marker on the right-most symbol of δ if it exists, and $\check{\Box}$ otherwise, i.e.,

$$\delta^< = \begin{cases} \zeta\check{d} & \text{if } \delta = \zeta d \quad (d \in \mathcal{D}_\Box, \zeta \in \mathcal{D}_\Box^*)\ , \text{ and} \\ \check{\Box} & \text{if } \zeta = \varepsilon\ . \end{cases}$$

(We use ε to denote the empty sequence.) Similarly, by ${}^>\delta$ we denote the element of $(\mathcal{D}_\Box \cup \check{\mathcal{D}_\Box})^*$ obtained by placing the tape head marker on the left-most symbol of δ if it exists, and $\check{\Box}$ otherwise, i.e.,

$${}^>\delta = \begin{cases} \check{d}\zeta & \text{if } \delta = d\zeta \quad (d \in \mathcal{D}_\Box, \zeta \in \mathcal{D}_\Box^*)\ , \text{ and} \\ \check{\Box} & \text{if } \zeta = \varepsilon\ . \end{cases}$$

Definition 2.5. *Let* $\mathcal{M} = (\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ *be an RTM. The* transition system $\mathcal{T}(\mathcal{M})$ associated with $\mathcal{M}$ *is defined as follows:*

1. *its set of states is the set of all configurations of* $\mathcal{M}$*;*

2. *its transition relation* $\rightarrow$ *is the least relation satisfying, for all* $a \in \mathcal{A}_\tau$, $d, e \in \mathcal{D}_\Box$ *and* $\delta_L, \delta_R \in \mathcal{D}_\Box^*$*:*

$$(s, \delta_L \check{d} \delta_R) \xrightarrow{a} (t, \delta_L{}^{<} e \delta_R) \text{ iff } s \xrightarrow{a[d/e]L} t \ \text{, and}$$

$$(s, \delta_L \check{d} \delta_R) \xrightarrow{a} (t, \delta_L e\, {}^{>} \delta_R) \text{ iff } s \xrightarrow{a[d/e]R} t \ \ ;$$

3. *its initial state is the configuration* $(\uparrow, \check{\Box})$*; and*

4. *its set of final states is the set of terminating configurations* $\{(s, \delta) \mid s\downarrow\}$*.*

Turing introduced his machines to define the notion of *effectively computable function*. By analogy, our notion of RTM can be used to define a notion of *effectively executable behaviour*.

Definition 2.6. *A transition system is* executable *if it is associated with an RTM.*

Parallel composition To illustrate how RTMs are suitable to model a form of interaction, we shall now define on RTMs a notion of parallel composition, equipped with a simple form communication. (We are not trying to define the most general or most suitable notion of parallel composition for RTMs here; the purpose of our notion of parallel composition is just to illustrate how RTMs may run in parallel and interact.) Let $\mathcal{C}$ be a finite set of *channels* for the communication of data symbols between one RTM and another, and let $\mathcal{A}' = \{c!d, c?d \mid c \in \mathcal{C}, d \in \mathcal{D}\}$; it is assumed that $\mathcal{A}' \subseteq \mathcal{A}$. Intuitively, $c!d$ stands for the action of sending datum d along channel c, while $c?d$ stands for the action of receiving datum d along channel c.

First, we define a notion of *parallel composition* on transition systems. Let $T_1 = (\mathcal{S}_1, \rightarrow_1, \uparrow_1, \downarrow_1)$ and $T_2 = (\mathcal{S}_2, \rightarrow_2, \uparrow_2, \downarrow_2)$ be transition systems; the *parallel composition* $[T_1 \parallel T_2]_\mathcal{C}$ of T_1 and T_2 is the transition system $[T_1 \parallel T_2]_\mathcal{C} = (\mathcal{S}, \rightarrow, \uparrow, \downarrow)$, with $\mathcal{S}$, $\rightarrow$, $\uparrow$ and $\downarrow$ defined by

1. $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$;

2. $(s_1, s_2) \xrightarrow{a} (s_1', s_2')$ iff $a \in \mathcal{A}_\tau - \mathcal{A}'$ and either
   (a) $s_1 \xrightarrow{a} s_1'$ and $s_2 = s_2'$, or $s_1 = s_1'$ and $s_2 \xrightarrow{a} s_2'$, or
   (b) $a = \tau$ and either $s_1 \xrightarrow{c!d} s_1'$ and $s_2 \xrightarrow{c?d} s_2'$, or $s_1 \xrightarrow{c?d} s_1'$ and $s_2 \xrightarrow{c!d} s_2'$ for some $c \in \mathcal{C}$ and $d \in \mathcal{D}$;

3. $\uparrow = (\uparrow_1, \uparrow_2)$; and

4. $\downarrow = \{(s_1, s_2) \mid s_1 \in \downarrow_1 \ \&\ s_2 \in \downarrow_2\}$.

Example 2.7. *Let* $\mathcal{A}$ *be as in Example 2.2. Figure 2 depicts the state-transition diagram of an RTM that enumerates the natural numbers and sends them along channel* i*. Let* $\mathcal{M}$ *denote the RTM specified in Figure 1, and let* $\mathcal{E}$ *denote the RTM specified in Figure 2. Then the parallel composition* $[\mathcal{M} \parallel \mathcal{E}]_i$ *exhibits the behaviour outputting, along channel* o*, all strings "*$1^n\#$*" with* n *even.*

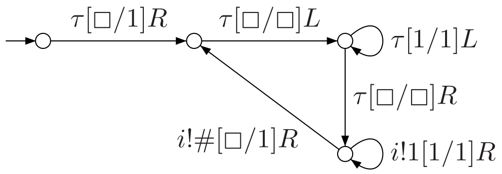

Figure 2: An RTM that enumerates and sends the strings 1#, 11#, 111#, ...

Equivalence In automata theory, Turing machines that compute the same function or accept the same language are generally considered equivalent. In fact, functional or language equivalence is underlying many of the standard notions and results in automata theory. Perhaps most notably, a *universal* Turing machine is a Turing machine that, when started with the code of some Turing machine on its tape, simulates this machine up to functional or language equivalence. A result from concurrency theory is that functional and language equivalence are arguably too coarse for reactive systems, because they abstract from all moments of choice. In concurrency theory many alternative behavioural equivalences have been proposed; we refer to [13] for a classification.

The results about RTMs that are obtained in the the remainder of this paper are modulo *branching bisimilarity* [15], which is the finest behavioural equivalence in van Glabbeek's linear time - branching time spectrum [13]. We shall consider both the divergence-insensitive and the divergence-preserving variant. (The divergence-preserving variant is called *branching bisimilarity with explicit divergence* in [15, 13], but in this paper we prefer the term *divergence-preserving* branching bisimilarity.)

We proceed to define the behavioural equivalences that we shall employ in this paper to compare transition systems. Let $\rightarrow$ be an $\mathcal{A}_\tau$-labelled transition relation on a set $\mathcal{S}$, and let $a \in \mathcal{A}_\tau$; we write $s \xrightarrow{(a)} t$ if $s \xrightarrow{a} t$ or $a = \tau$ and $s = t$. Furthermore, we denote the transitive closure of $\xrightarrow{\tau}$ by $\longrightarrow^+$, and we denote the reflexive-transitive closure of $\xrightarrow{\tau}$ by $\twoheadrightarrow$.

Definition 2.8. *Let $T_1 = (\mathcal{S}_1, \rightarrow_1, \uparrow_1, \downarrow_1)$ and $T_2 = (\mathcal{S}_2, \rightarrow_2, \uparrow_2, \downarrow_2)$ be transition systems. A* branching bisimulation *from T_1 to T_2 is a binary relation $\mathcal{R} \subseteq \mathcal{S}_1 \times \mathcal{S}_2$ and, for all states s_1 and s_2, $s_1 \mathrel{\mathcal{R}} s_2$ implies*

1. *if $s_1 \xrightarrow{a}_1 s_1'$, then there exist $s_2', s_2'' \in \mathcal{S}_2$ such that $s_2 \twoheadrightarrow_2 s_2'' \xrightarrow{(a)}_2 s_2'$, $s_1 \mathrel{\mathcal{R}} s_2''$ and $s_1' \mathrel{\mathcal{R}} s_2'$;*
2. *if $s_2 \xrightarrow{a}_2 s_2'$, then there exist $s_1', s_1'' \in \mathcal{S}_1$ such that $s_1 \twoheadrightarrow_1 s_1'' \xrightarrow{(a)}_1 s_1'$, $s_1'' \mathrel{\mathcal{R}} s_2$ and $s_1' \mathrel{\mathcal{R}} s_2'$;*
3. *if $s_1\downarrow_1$, then there exists s_2' such that $s_2 \twoheadrightarrow_2 s_2'$, $s_1 \mathrel{\mathcal{R}} s_2'$ and $s_2'\downarrow_2$; and*
4. *if $s_2\downarrow_2$, then there exists s_1' such that $s_1 \twoheadrightarrow_1 s_1'$, $s_1' \mathrel{\mathcal{R}} s_2$ and $s_1'\downarrow_1$.*

The transition systems T_1 and T_2 are branching bisimilar *(notation: $T_1 \leftrightarroweq_b T_2$) if there exists a branching bisimulation from T_1 to T_2 such that $\uparrow_1 \mathrel{\mathcal{R}} \uparrow_2$.*

A branching bisimulation $\mathcal{R}$ from T_1 to T_2 is divergence-preserving *if, for all states s_1 and s_2, $s_1 \mathrel{\mathcal{R}} s_2$ implies*

5. *if there exists an infinite sequence $(s_{1,i})_{i\in\mathbb{N}}$ such that $s_1 = s_{1,0}$, $s_{1,i} \xrightarrow{\tau} s_{1,i+1}$ and $s_{1,i} \mathrel{\mathcal{R}} s_2$ for all $i \in \mathbb{N}$, then there exists a state s_2' such that $s_2 \longrightarrow^+ s_2'$ and $s_{1,i} \mathrel{\mathcal{R}} s_2'$ for some $i \in \mathbb{N}$; and*
6. *if there exists an infinite sequence $(s_{2,i})_{i\in\mathbb{N}}$ such that $s_2 = s_{2,0}$, $s_{2,i} \xrightarrow{\tau} s_{2,i+1}$ and $s_1 \mathrel{\mathcal{R}} s_{2,i}$ for all $i \in \mathbb{N}$, then there exists a state s_1' such that $s_1 \longrightarrow^+ s_1'$ and $s_1' \mathrel{\mathcal{R}} s_{2,i}$ for some $i \in \mathbb{N}$.*

The transition systems T_1 *and* T_2 *are* divergence-preserving branching bisimilar *(notation:* $T_1 \leftrightarrow_b^\Delta T_2$*) if there exists a divergence-preserving branching bisimulation from* T_1 *to* T_2 *such that* $\uparrow_1 \mathcal{R} \uparrow_2$*.*

The notions of branching bisimilarity and divergence-preserving branching bisimilarity originate with [15]. The particular divergence conditions we use to define divergence-preserving branching bisimulations here are discussed in [14], where it is also proved that divergence-preserving branching bisimilarity is an equivalence.

Definition 2.9. *Let* T *be a transition system and let* s *and* t *be two states in* T*. A* τ*-transition* $s \xrightarrow{\tau} t$ *is* inert *if* s *and* t *are related by the maximal divergence-preserving branching bisimulation on* T*.*

If s and t are distinct states, then an inert τ-transition $s \xrightarrow{\tau} t$ can be *eliminated* from a transition system, e.g., by removing all outgoing transitions of s, changing every outgoing transition $t \xrightarrow{a} u$ from t to an outgoing transition $s \xrightarrow{a} u$, and removing the state t. This operation yields a transition system that is divergence-preserving branching bisimilar to the original transition system.

An unobservable transition of an RTM, i.e., a transition labelled with τ, may be thought of as an internal computation step. Divergence-preserving branching bisimilarity allows us to abstract from internal computations as long as they do not discard the option to execute a certain behaviour. The following notion of will be technically convenient in the remainder of the paper.

Definition 2.10. *Given some transition system* T*, an* internal computation from state s to s' *is a sequence of states* $s_1, \ldots, s_n$ *in* T *such that* $s = s_1 \xrightarrow{\tau} \ldots \xrightarrow{\tau} s_n = s'$*. An internal computation is called* fully deterministic *iff, for every state* s_i *(*$1 \leq i < n$*),* $s_i \xrightarrow{a} s_i'$ *implies* $a = \tau$ *and* $s_i' = s_{i+1}$*. We shall write* $s \twoheadrightarrow s'$ *if there exists a fully deterministic computation from* s *to* s'*.*

Lemma 2.11. *Let* T *be a transition system and let* s *and* t *be two states in* T*. If* $s \twoheadrightarrow s'$*, then* s *and* s' *are related by the maximal divergence-preserving branching bisimulation on* T*.*

3 Expressiveness of RTMs

To confirm the expressiveness of RTMs, we shall establish in this section that every effective transition system can be simulated up to branching bisimilarity, and that every computable transition system can be simulated up to divergence-preserving branching bisimilarity. We use this as an auxiliary result to establish that a parallel composition of RTMs can be simulated by a single RTM, and we derive from it the existence of universal RTMs.

Let $T = (\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ be a transition system; the mapping $out(\_) : \mathcal{S} \rightarrow 2^{\mathcal{A}_\tau \times \mathcal{S}}$ associates with every state its set of outgoing transitions, i.e., for all $s \in \mathcal{S}$,

$$out(s) = \{(a, t) \mid s \xrightarrow{a} t\} \ ;$$

and we denote by $fin(\_)$ the characteristic function of $\downarrow$.

Definition 3.1. *Let* $T = (\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ *be an* $\mathcal{A}_\tau$*-labelled transition system. We say that* T *is* effective *if* $\rightarrow$ *and* $\downarrow$ *are recursively enumerable. We say that* T *is* computable *if both the functions* $out(\_)$ *and* $fin(\_)$ *are recursive.*

We shall, in this paper, not go into the details of explaining more carefully what are suitable codings into natural numbers of $\mathcal{A}_\tau$ and $\mathcal{S}$, and how they should be extended to codings of $\rightarrow$, $\downarrow$, $out(\_)$ and $fin(\_)$ so that the formal theory of

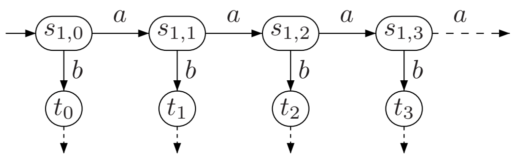

Figure 3: The transition system T_1.

recursiveness makes sense for arbitrary (countable) transition systems. (The reader may want to consult [21, §1.10] for more explanations.) If $\rightarrow$ and $\downarrow$ are recursively enumerable, then this, intuitively, means that there exist algorithms that enumerate the transitions in $\rightarrow$ and the states in $\downarrow$. If the function $out(\_)$ is recursive, then there exists an algorithm that, given a state s, yields the list of outgoing transitions, and if the function $fin(\_)$ is recursive, then there exists an algorithm that, given a state s, determines if $s \in \downarrow$.

Proposition 3.2. *The transition system associated with an RTM is computable.*

Phillips proved in [19] that every effective transition system is *weakly bisimilar* to a computable transition system. It can easily be seen that his proof actually associates with every effective transition system a *branching bisimilar* computable transition system of which, moreover, every state has a branching degree of at most 2.

Definition 3.3. *Let $T = (\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ be a transition system, and let B be a natural number. We say that T has a branching degree* bounded by B *if, for every state $s \in \mathcal{S}$, $|out(s)| \leq B$. We say that T is* boundedly branching *if there exists $B \in \mathbb{N}$ such that the branching degree of T is bounded by B.*

Proposition 3.4 (Phillips)**.** *For every effective transition system T there exists a boundedly branching computable transition system T' such that $T \leftrightarrow_b T'$.*

A crucial insight in Phillips' proof is that a divergence (i.e., an infinite sequence of τ-transitions) can be exploited to simulate a state of which the set of outgoing transitions is recursively enumerable, but not recursive. The following example shows that introducing divergence is unavoidable.

Example 3.5. *(In this example, we denote by φ_x the partial recursive function with index $x \in \mathbb{N}$ in some exhaustive enumeration of partial recursive functions, see, e.g., [21].) Assume that $\mathcal{A} = \{a, b\}$, and consider the transition system $T_1 = (\mathcal{S}_1, \rightarrow_1, \uparrow_1, \downarrow_1)$ with $\mathcal{S}_1$, $\rightarrow_1$, $\uparrow_1$ and $\downarrow_1$ defined by*

$$\begin{aligned} &\mathcal{S}_1 = \{s_{1,x}, t_{1,x} \mid x \in \mathbb{N}\} \ , \\ &\rightarrow_1 = \{(s_{1,x}, a, s_{1,x+1}) \mid x \in \mathbb{N}\} \cup \{(s_{1,x}, b, t_{1,x}) \mid x \in \mathbb{N}\} \ , \\ &\uparrow_1 = s_{1,0} \ , \ \textit{and} \\ &\downarrow_1 = \{t_{1,x} \mid \varphi_x(x) \ \textit{converges}\} \ . \end{aligned}$$

(See also Figure 3).

Now suppose that T_2 is a transition system such that $T_1 \leftrightarrow_b^{\Delta} T_2$, as witnessed by some divergence-preserving branching bisimulation relation $\mathcal{R}$; we argue that T_2 is not computable by deriving a contradiction from the assumption that it is.

Clearly, since T_1 does not admit infinite sequences of τ-transitions, if $\mathcal{R}$ is divergence-preserving, then T_2 does not admit infinite sequences of τ-transitions either. It follows that if $s_1 \ \mathcal{R} \ s_2$, then there exists a state s_2' in T_2 such that $s_2 \longrightarrow\!\!\!\!\rightarrow_2 s_2$, $s_1 \ \mathcal{R} \ s_2'$, and $s_2' \xrightarrow{\tau}\!\!\!\!\!/\ $. Moreover, since T_2 is computable and does not *admit infinite sequences of consecutive τ-transitions, a state s_2' satisfying the aforementioned properties is produced by the algorithm that, given a state of T_2, selects an enabled τ-transition and recurses on the target of the transition until it reaches a state in which no τ-transitions are enabled.*

But then we also have an algorithm that determines if $\varphi_x(x)$ converges:

1. *it starts from the initial state $\uparrow_2$ of T_2;*
2. *it runs the algorithm to find a state without outgoing τ-transitions, and then it repeats the following steps x times:*
   - (a) *execute the inc-transition enabled in the reached state;*
   - (b) *run the algorithm to find a state without outgoing τ-transitions again;*

   since $\uparrow_1 \mathcal{R} \uparrow_2$, this yields a state $s_{2,x}$ in T_2 such that $s_{1,x} \mathcal{R} s_{2,x}$;
3. *it executes the run-transition that must be enabled in $s_{2,x}$, followed, again, by the algorithm to find a state without outgoing τ-transitions; this yields a state $t_{2,x}$, without any outgoing transitions, such that $t_{1,x} \mathcal{R} t_{2,x}$.*

From $t_{1,x} \mathcal{R} t_{2,x}$ it follows that $t_{2,x} \in \downarrow_2$ iff $\varphi_x(x)$ converges, so the problem of deciding whether $\varphi_x(x)$ converges has been reduced to the problem of deciding whether $t_{2,x} \in \downarrow_2$. Since it is undecidable if $\varphi_x(x)$ converges, it follows that $\downarrow_2$ is not recursive, which contradicts our assumption that T_2 is computable.

3.1 Simulating Boundedly Branching Computable Transition Systems

By Proposition 3.4, in order to prove that every effective transition systems can be simulated up to branching bisimilarity by an RTM, it suffices to prove that every boundedly branching computable transition system can be simulated by an RTM. We shall now proceed to prove this, and, in fact, we shall establish that the simulation can be made to preserve divergence.

For the remainder of this section let $T = (\mathcal{S}_T, \rightarrow_T, \uparrow_T, \downarrow_T)$ be a boundedly branching computable transition system, say with branching degree bounded by B. We shall construct an RTM $\mathcal{M} = (\mathcal{S}_\mathcal{M}, \rightarrow_\mathcal{M}, \uparrow_\mathcal{M}, \downarrow_\mathcal{M})$, called the *simulator* for T, such that $\mathcal{T}(\mathcal{M}) \leftrightarrows_b^\Delta T$.

Tape. Let us assume encodings of the functions $\ulcorner \_ \urcorner : out(\_) \rightarrow \mathbb{N}$, $\ulcorner \_ \urcorner : fin(\_) \rightarrow \mathbb{N}$, and the sets $\ulcorner \_ \urcorner : \mathcal{A}_\tau \rightarrow \{1, \ldots, |\mathcal{A}_\tau|\}$ and $\ulcorner \_ \urcorner : \mathcal{S}_\mathcal{M} \rightarrow \mathbb{N}$; the simulator RTM $\mathcal{M}$ stores these functions, actions, states and transitions on its tape as natural numbers. The existence of the encodings of the functions $out(\_)$ and $fin(\_)$ is due the the fact that they are recursive.

The way in which natural numbers are represented as sequences over some finite alphabet of tape symbols is largely irrelevant, but in our construction below it is sometimes convenient to have an explicit representation. In such cases, we assume that numbers are stored in unary notation using the symbol 1. That is, a natural number n is represented on the tape as the sequence 1^{n+1} of $n + 1$ occurrences of the symbol 1. In addition to the symbol 1, we use the symbols $\llbracket$ and $\rrbracket$ to enclose the (static) codes of the two functions that steer the simulation of T on the tape, $|$ to separate the elements of a tuple of natural numbers, and # to separate tuples. The RTM $\mathcal{M}$ constructed below will incorporate the operation of some auxiliary Turing machines that may use some extra encoding and symbols; let $\mathcal{D}'$ be the collection of all these extra symbols. Then the tape alphabet $\mathcal{D}$ of $\mathcal{M}$ is

$$\mathcal{D} = \{1, \llbracket, \rrbracket, |, \#\} \cup \mathcal{D}' \ .$$

We shall define $\mathcal{M}$ as the union of three fragments: an *initialisation fragment*, a *state fragment*, and a *step fragment*. Instead of directly using (conventional) Turing machines computing $out(\_)$ and $fin(\_)$ we store their codes on the tape and use a Turing machine to interpret them these codes. This is slightly more generic than necessary; the advantage of proceeding in this way is that we can easily adapt the simulator to obtain a universal RTM (in Sect. 3.3).

Initialisation fragment. The *initialisation fragment* Init prepares the tape for simulation of T by first writing the symbol $[\![$ on the tape, followed by (the codes of) the functions $out(\_)$ and $fin(\_)$ belonging to T which are separated by the symbol $|$. Then it writes the symbol $]\!]$ on the tape followed by the code of the initial state of T. Thereafter, it returns the tape head to the symbol $]\!]$. Let $\mathcal{M}_i$ be an RTM that achieves precisely this; when started with an empty tape ($\check{\Box}$), it halts with the tape instance $[\![\ulcorner out \urcorner | \ulcorner fin \urcorner \check{]\!]} \ulcorner \uparrow_T \urcorner$.

The set of states of Init is defined as

$$\mathcal{S}_{\mathsf{Init}} = \mathcal{S}_{\mathcal{M}_i} \setminus \downarrow_{\mathcal{M}_i} \ ,$$

its initial state is defined as

$$\uparrow_{\mathsf{Init}} = \uparrow_{\mathcal{M}_i} \ ; \text{ and}$$

its set of transitions is defined as

$$\begin{aligned}\rightarrow_{\mathsf{Init}} = \{&(in, d, \tau, e, M, in') \mid (in, d, \tau, e, M, in') \in \rightarrow_{\mathcal{M}_i},\ in' \in \mathcal{S}_{\mathcal{M}_i} \setminus \downarrow_{\mathcal{M}_i}\}\}\\ &\cup \{(in, d, \tau, e, M, \uparrow_{\mathsf{State}}) \mid (in, d, \tau, e, M, in') \in \rightarrow_{\mathcal{M}_i},\ in' \in \downarrow_{\mathcal{M}_i}\}\ .\end{aligned}$$

Lemma 3.6. *The fragment* Init *has a deterministic computation from* $(\uparrow_{\mathsf{Init}}, \check{\Box})$ *to* $(\uparrow_{\mathsf{State}}, [\![\ulcorner out \urcorner | \ulcorner fin \urcorner \check{]\!]} \ulcorner \uparrow_T \urcorner)$.

State fragment. The *state fragment* State replaces the code of the current state on the tape by a sequence of codes that represents the behaviour of T in the current state. It is assumed that it starts with a tape instance of the form $[\![\ulcorner out \urcorner | \ulcorner fin \urcorner \check{]\!]} \ulcorner s \urcorner$ with $s \in \mathcal{S}_T$.

Recall that the functions $out(\_)$ and $fin(\_)$ are both recursive. Hence, by [21] there exists a (conventional) deterministic Turing machine $\mathcal{M}_s$ that, when it is started with a tape instance $[\![\ulcorner out \urcorner | \ulcorner fin \urcorner \check{]\!]} \ulcorner s \urcorner$ terminates with the tape instance

$$[\![\ulcorner out \urcorner | \ulcorner fin \urcorner \check{]\!]} \ulcorner (s \in \downarrow_T)? \urcorner | \ulcorner a_1 \urcorner | \cdots | \ulcorner a_k \urcorner \# \ulcorner s_1 \urcorner | \cdots | \ulcorner s_k \urcorner \#\ ,$$

where $out(s) = \{(a_i, s_i) \mid 1 \le i \le k\}$ and $\ulcorner (s \in \downarrow_T)? \urcorner$ is a special code denoting $fin(s)$. Note that, since the branching degree of T is bounded by B, we have that $k \le B$. We assume without loss of generality that the Turing machine $\mathcal{M}_s$ first copies the codes of $out(\_)$ and $fin(\_)$ to the right of the symbol $]\!]$ and thereafter never crosses this boundary symbol again for its computation. We refer to the sequence $(s \in \downarrow_T)?, a_1, \ldots, a_k$ that is generated and stored on the tape by $\mathcal{M}_s$ as the *menu* in s.

The set of states of State is defined as

$$\mathcal{S}_{\mathsf{State}} = \mathcal{S}_{\mathcal{M}_s} \setminus \downarrow_{\mathcal{M}_s} \ ;$$

its initial states is defined as

$$\uparrow_{\mathsf{State}} = \uparrow_{\mathcal{M}_s} \ ; \text{ and}$$

its set of transitions is defined as

$$\begin{aligned}\rightarrow_{\mathsf{State}} = \{(st, d, \tau, e, M, st') \mid (st, d, e, M, st') \in \rightarrow_{\mathcal{M}_s},\ st' \in \mathcal{S}_{\mathsf{State}} \setminus \downarrow_{\mathcal{M}_s}\} \\ \cup \{(st, d, \tau, e, M, \uparrow_{\mathsf{Step}}) \mid (st, d, e, M, st') \in \rightarrow_{\mathcal{M}_s},\ st' \in \downarrow_{\mathcal{M}_s}\}\ .\end{aligned}$$

(Note how we associate with $\mathcal{M}_s$ (a fragment of) an RTM by adding τ-labels to its transitions.)

Lemma 3.7. *The fragment* State *has a deterministic computation from configuration* $(\uparrow_{\mathsf{State}}, \llbracket \ulcorner out \urcorner | \ulcorner fin \urcorner \check{\rrbracket} \ulcorner s \urcorner)$ *for each* $s \in \mathcal{S}_T$ *to*

$$(\uparrow_{\mathsf{Step}}, \llbracket \ulcorner out \urcorner | \ulcorner fin \urcorner \check{\rrbracket} \ulcorner (s \in \downarrow_T)? \urcorner | \ulcorner a_1 \urcorner | \cdots | \ulcorner a_k \urcorner \# \ulcorner s_1 \urcorner | \cdots | \ulcorner s_k \urcorner \#)\ ,$$

where the part to the right of the symbol $\rrbracket$ *on the tape represents the menu generated by a applying the functions* $out(\_)$ *and* $fin(\_)$ *to* s*.*

Step fragment. The purpose of the *step fragment* Step is to select an action a_i from the set of enabled actions in the current state, execute that action, and remove $\ulcorner (s \in \downarrow_T)? \urcorner$ and all (codes of) actions and states from the tape, except the code of the target state of the a_i-transition.

The state s in the simulated transition system T embodies a choice between its k outgoing transitions $s \xrightarrow{a_1} s_1, \ldots, s \xrightarrow{a_k} s_k$, and is terminating if, and only if, $s \in \downarrow_T$. In order to get a branching bisimulation between T and the transition system associated with $\mathcal{M}$, the latter will necessarily have to include a configuration offering the same choice of outgoing transitions and the same termination behaviour. It is important to note that branching bisimilarity does not, e.g., allow the choice for one of the outgoing transitions to be made by a computation (resulting in a sequence of τ-transitions) that eliminates options one by one. The fragment Step will therefore have to include a special state $sp_{(s \in \downarrow_T)?, a_1, \ldots, a_k}$, for every potential menu. (Note that, since $k \leq B$, there will be at most $N = \sum_{k=0}^{B} 2 \cdot |\mathcal{A}_\tau|^k$ different menus in T.)

The functionality of the step fragment is split up in two parts: before and after the simulation of an a_i-transition. The first part uses the RTM $\mathcal{M}_{pd}$ to decode the menu on the tape ending up in the state $sp_{(s \in \downarrow_T)?, a_1, \ldots, a_k}$ from which termination, if enabled, or an a_i-transition can occur. In case the transition is performed, the second part finds the target state s_i of the a_i-transition. The RTM $\mathcal{M}_{pm}$ will move the code $\ulcorner s_i \urcorner$ to the right of the symbol $\rrbracket$ and the RTM $\mathcal{M}_{pc}$ will empty the remaining part of the tape.

The fragment Step starts from a tape instance of the form

$$\llbracket \ulcorner out \urcorner | \ulcorner fin \urcorner \check{\rrbracket} \ulcorner (s \in \downarrow_T)? \urcorner | \ulcorner a_1 \urcorner | \cdots | \ulcorner a_k \urcorner \# \ulcorner s_1 \urcorner | \cdots | \ulcorner s_k \urcorner \#$$

and then progresses to the state $sp_{(s \in \downarrow_T)?, a_1, \ldots, a_k}$, while removing the symbols $\ulcorner (s \in \downarrow_T)? \urcorner | \ulcorner a_1 \urcorner | \cdots | \ulcorner a_k \urcorner$ from the tape; this is a matter of decoding the information on the tape. For this decoding part we assume that $\mathcal{M}_{pd}$ is an RTM that halts with the tape instance $\llbracket \ulcorner out \urcorner | \ulcorner fin \urcorner \rrbracket \Box \cdots \Box \check{\#} \ulcorner s_1 \urcorner | \cdots | \ulcorner s_k \urcorner \#$. Among the states of $\mathcal{M}_{pd}$ we have the previously mentioned special states $sp_{(s \in \downarrow_T)?, a_1, \ldots, a_k}$ for all $(s \in \downarrow_T?) \in \{yes, no\}, a_1, \ldots, a_k \in \mathcal{A}_\tau, k \leq B$. A state $sp_{(s \in \downarrow_T)?, a_1, \ldots, a_k}$ is declared final if, and only if, $s \in \downarrow_T$, and it has an outgoing a_i-transitions to the states ne_i $(1 \leq i \leq k)$.

After the decoding part, the action a_i can be performed (while removing the symbol #) and the fragment ends up in the state ne_i. The goal of the states ne_i down to ne_1 is to find the code $\ulcorner s_i \urcorner$, replacing the symbols preceding $\ulcorner s_i \urcorner$ by $\Box$, and to yield the tape instance $\llbracket \ulcorner out \urcorner | \ulcorner fin \urcorner \rrbracket \Box \cdots \Box \, {}^{>}\ulcorner s_i \urcorner | \ldots | \ulcorner s_k \urcorner \#$.

Let $\mathcal{M}_{pm}$ be an RTM that, when started with above tape instance, moves the found state code $\ulcorner s_i \urcorner$ to the right of the symbol $\rrbracket$ and halts with the tape instance $\llbracket \ulcorner out \urcorner | \ulcorner fin \urcorner \rrbracket \ulcorner s_i \urcorner \square \cdots \square \check{|} \ulcorner s_{i+1} \urcorner | \ldots | \ulcorner s_k \urcorner \#$.

Then, let $\mathcal{M}_{pc}$ be an RTM that, when started with the above tape instance, empties the remaining part of the tape, moves the tape head back to the symbol $\rrbracket$ and halts with the tape instance $\llbracket \ulcorner out \urcorner | \ulcorner fin \urcorner \check{\rrbracket} \ulcorner s_i \urcorner$.

The set of states of Step is defined as

$$\mathcal{S}_{\mathsf{Step}} = (\mathcal{S}_{\mathcal{M}_{pd}} \cup \{ne_1, \ldots, ne_B\} \cup \mathcal{S}_{\mathcal{M}_{pm}} \cup \mathcal{S}_{\mathcal{M}_{pc}}) \setminus (\downarrow_{\mathcal{M}_{pd}} \cup \downarrow_{\mathcal{M}_{pm}} \cup \downarrow_{\mathcal{M}_{pc}}) \ ;$$

its initial states is defined as

$$\uparrow_{\mathsf{Step}} = \uparrow_{\mathcal{M}_{pd}} \ ; \text{ and}$$

its set of transitions is defined as

$$\begin{aligned}
\rightarrow_{\mathsf{Step}} = \{&(sp, d, \tau, e, M, sp') \mid (sp, d, \tau, e, M, sp') \in \rightarrow_{\mathcal{M}_{pd}}\} \\
\cup \{&(sp_{(s \in \downarrow_T), a_1, \ldots, a_k}, \#, a_i, \square, R, ne_i) \\
&\quad \mid (s \in \mathcal{S}_T)? \in \{yes, no\}, a_1, \ldots, a_k \in \mathcal{A}_\tau, k \leq B, 1 \leq i \leq k\} \\
\cup \{&(ne_k, 1, \tau, \square, R, ne_k), (ne_k, |, \tau, \square, R, ne_{k-1}) \mid 1 < k \leq B\} \\
\cup \{&(ne_1, d, \tau, e, M, sp') \mid (\uparrow_{\mathcal{M}_{pm}}, d, \tau, e, M, sp') \in \rightarrow_{\mathcal{M}_{pm}}\} \\
\cup \{&(sp, d, \tau, e, M, sp') \mid (sp, d, \tau, e, M, sp') \in \rightarrow_{\mathcal{M}_{pm}},\ sp' \in \mathcal{S}_{\mathcal{M}_{pm}} \setminus \downarrow_{\mathcal{M}_{pm}}\} \\
\cup \{&(sp, d, \tau, e, M, \uparrow_{\mathcal{M}_{pc}}) \mid (sp, d, \tau, e, M, sp') \in \rightarrow_{\mathcal{M}_{pm}},\ sp' \in \downarrow_{\mathcal{M}_{pm}}\} \\
\cup \{&(sp, d, \tau, e, M, sp') \mid (sp, d, \tau, e, M, sp') \in \rightarrow_{\mathcal{M}_{pc}},\ sp' \in \mathcal{S}_{\mathcal{M}_{pc}} \setminus \downarrow_{\mathcal{M}_{pc}}\} \\
\cup \{&(sp, d, \tau, e, M, \uparrow_{\mathsf{State}}) \mid (sp, d, \tau, e, M, sp') \in \rightarrow_{\mathcal{M}_{pc}},\ sp' \in \downarrow_{\mathcal{M}_{pc}}\} \ .
\end{aligned}$$

See Figure 4 for a schematic overview of the fragment Step. Note that in this figure—for clarity reasons—only one of possibly many states $sp_{(s \in \downarrow)?, a_1, \ldots, a_k}$ and transition thereto is drawn.

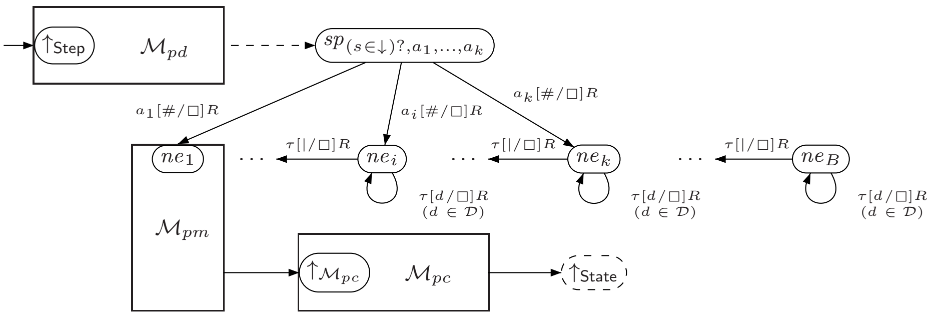

Figure 4: Diagram of the step fragment.

As mentioned before we can split the fragment up in two parts; we obtain the following two lemmas.

Lemma 3.8. *The fragment* Step *(using the auxiliary RTM $\mathcal{M}_{pd}$) has a deterministic computation from*

$$(\uparrow_{\mathsf{Step}}, \llbracket \ulcorner out \urcorner | \ulcorner fin \urcorner \check{\rrbracket} \ulcorner (s \in \downarrow_T)? \urcorner | \ulcorner a_1 \urcorner | \cdots | \ulcorner a_k \urcorner \# \ulcorner s_1 \urcorner | \cdots | \ulcorner s_k \urcorner \#)$$

to $(sp_{(s \in \downarrow_T)?, a_1, \ldots, a_k}, \llbracket \ulcorner out \urcorner | \ulcorner fin \urcorner \rrbracket \square \cdots \square \check{\#} \ulcorner s_1 \urcorner | \cdots | \ulcorner s_k \urcorner \#)$.

Lemma 3.9. *The fragment* Step *(using the auxiliary RTMs $\mathcal{M}_{pm}$ and $\mathcal{M}_{pc}$) has a deterministic computation from* $(ne_i, \llbracket \ulcorner out \urcorner | \ulcorner fin \urcorner \rrbracket \square \cdots \square \,{}^{>}\ulcorner s_1 \urcorner | \cdots | \ulcorner s_k \urcorner \#)$ *to* $(\uparrow_{\mathsf{State}}, \llbracket \ulcorner out \urcorner | \ulcorner fin \urcorner \check{\rrbracket} \ulcorner s_i \urcorner)$.

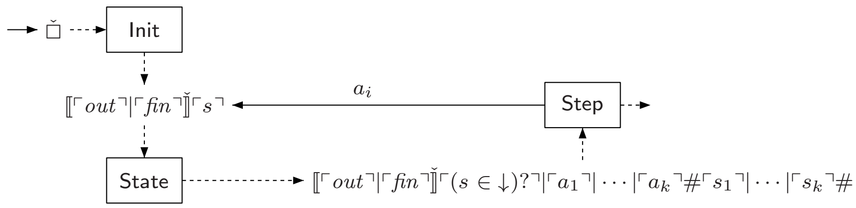

Figure 5: Diagram of the deterministic computable transition system simulator.

Simulator. The *simulator RTM* $\mathcal{M} = (\mathcal{S}_{\mathcal{M}}, \rightarrow_{\mathcal{M}}, \uparrow_{\mathcal{M}}, \downarrow_{\mathcal{M}})$ is defined as the combination of the fragments Init, State and Step defined above. The set of states of $\mathcal{M}$ is defined as the union of the sets of states of all fragments:

$$\mathcal{S}_{\mathcal{M}} = \mathcal{S}_{\mathsf{Init}} \cup \mathcal{S}_{\mathsf{State}} \cup \mathcal{S}_{\mathsf{Step}} \ \ ;$$

the transition relation of $\mathcal{M}$ is the union of the transition relations of all fragments:

$$\mathcal{S}_{\mathcal{M}} = \rightarrow_{\mathsf{Init}} \cup \rightarrow_{\mathsf{State}} \cup \rightarrow_{\mathsf{Step}} \ \ ;$$

the initial state of $\mathcal{M}$ is the initial state of Init:

$$\uparrow_{\mathcal{M}} = \uparrow_{\mathsf{Init}} \ \ ; \text{ and}$$

the set of final states of $\mathcal{M}$ consists of the states of Step $sp_{(s \in \downarrow_T)?, a_1, \ldots, a_k}$ where s is a final state in T

$$\downarrow_{\mathcal{M}} = \{ sp_{(s \in \downarrow_T)?, a_1, \ldots, a_k} \mid s \in \downarrow_T \} \ \ .$$

Fig. 5 schematically illustrates how the fragments are combined to constitute the simulator $\mathcal{M}$.

Theorem 3.10. *For every boundedly branching computable transition system T there exists a reactive Turing machine $\mathcal{M}$ such that $\mathcal{T}(\mathcal{M}) \leftrightarrow_b^{\Delta} T$.*

Proof. Consider the RTM $\mathcal{M}$ of which the definition is sketched above. Using Lemma 3.6 we define the following relation:

$$\mathcal{R}_{\uparrow} = \{ (\uparrow_T, t) \mid t \in \{ (\uparrow_{\mathsf{Init}}, \check{\Box}), \ldots, (\uparrow_{\mathsf{State}}, [\![\ulcorner out \urcorner | \ulcorner fin \urcorner \check{]\!]} \ulcorner \uparrow_T \urcorner) \} \} \ \ .$$

Using Lemmas 3.9, 3.7 and 3.8 we define the following relation for each $s \in \mathcal{S}_T$:

$$\mathcal{R}_s = \{ (s, t) \mid t \in \langle (ne_i, [\![\ulcorner out \urcorner | \ulcorner fin \urcorner]\!] \Box \cdots \Box \overset{>}{\ulcorner} s_1 \urcorner | \cdots | \ulcorner s_k \urcorner \#), \ldots, (sp_{(s \in \downarrow_T)?, a_1, \ldots, a_k}, [\![\ulcorner out \urcorner | \ulcorner fin \urcorner]\!] \Box \cdots \Box \check{\#} \ulcorner s_1 \urcorner | \cdots | \ulcorner s_k \urcorner \#) \rangle \} \ \ .$$

We can now define the relation

$$\mathcal{R} = \mathcal{R}_{\uparrow} \cup \bigcup_{s \in \mathcal{S}_T} \mathcal{R}_s \ \ .$$

The relation $\mathcal{R}$ is a divergence-preserving branching bisimulation between $\mathcal{T}(\mathcal{M})$ and T. □

Combining the above theorem with Proposition 3.4 we can conclude that reactive Turing machines can simulate effective transition systems up to branching bisimilarity, but, in view of Example 3.5, not in a divergence-preserving manner.

Corollary 3.11. *For every effective transition system T there exists a reactive Turing machine $\mathcal{M}$ such that $\mathcal{T}(\mathcal{M}) \leftrightarrow_b T$.*

Note that all computations involved in the simulation of T are deterministic (see Lemmas 3.6–3.9). If $\mathcal{M}$ is non-deterministic, then this is due to some state $sp_{(s\in\downarrow)?,a_1,\ldots,a_k}$ with some action a occurs more than once in the sequence $a_1,\ldots,a_k$. It follows that a deterministic computable transition system can be simulated up to divergence-preserving branching bisimilarity by a deterministic reactive Turing machine. We proceed to formally state this result below.

Definition 3.12. *A transition system $T = (\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ is* deterministic *if, for every state $s \in \mathcal{S}$ and for every $a \in \mathcal{A}_\tau$, $s \xrightarrow{a} s_1$ and $s \xrightarrow{a} s_2$ implies $s_1 = s_2$.*

Clearly, if T is deterministic, then, for every state s in T, $|out(s)| \leq |\mathcal{A}_\tau|$. So a deterministic transition system is boundedly branching, and therefore we get the following corollary to Theorem 3.10.

Corollary 3.13. *For every deterministic computable transition system T there exists a deterministic reactive Turing machine $\mathcal{M}$ such that $\mathcal{T}(\mathcal{M}) \leftrightarroweq_b^\Delta T$.*

3.2 Parallel Composition

Using Theorem 3.10 we can now also establish that a parallel composition of RTMs can be simulated, up to divergence-preserving branching bisimilarity, by a single RTM. To this end, note that the transition systems associated with RTMs are boundedly branching and computable. Further note that the parallel composition of boundedly branching computable transition systems is again computable. It follows that the transition system associated with a parallel composition of RTMs is boundedly branching and computable, and hence, by Theorem 3.10, there exists an RTM that simulates this transition system up to divergence-preserving branching bisimilarity. Thus we get the following corollary.

Corollary 3.14. *For every pair of reactive Turing machines $\mathcal{M}_1$ and $\mathcal{M}_2$ and for every set of communication channels $\mathcal{C}$ there exists an RTM $\mathcal{M}$ such that $\mathcal{T}(\mathcal{M}) \leftrightarroweq_b^\Delta \mathcal{T}([\mathcal{M}_1 \parallel \mathcal{M}_2]_\mathcal{C})$.*

3.3 Universality

In the theory of computation a classical and central notion is the *universal Turing machine*: a Turing machine that can simulate any arbitrary Turing machine on arbitrary input. Here, the (encoded) description of a Turing machine and the input are present on the tape beforehand. In this subsection we propose a notion of universal RTM and investigate to what extent such universal RTMs exist. Naturally, our notion of universal RTM should reflect our desiderata for introducing RTMs:

Firstly, since our main aim is to formalise communication explicitly, we want a universal RTM to first receive input via communication rather than finding it on its tape at the beginning of its operation (recall our assumption that the tape of our RTM is initially empty). To this end, we associate with the encoding $\ulcorner\mathcal{M}\urcorner$ of some RTM $\mathcal{M}$ (see [21]) an RTM $\overline{\mathcal{M}}$ that sends $\ulcorner\mathcal{M}\urcorner$ along channel u and then terminates. This RTM $\overline{\mathcal{M}}$ will be put in parallel with the universal RTM to be defined, abstracting from communication over the channel u.

Secondly, the simulation of other Turing machines by a universal Turing machine is in the classical theory up to language equivalence. For example, Hopcroft, Motwani and Ullman define in [18] the universal Turing machine for the so-called universal language. Language equivalence is, however, too coarse if one is interested in the behaviour of an RTM rather than only the function it computes. Our notion of universal RTM should simulate every RTM up to divergence-preserving branching bisimulation instead of language equivalence.

An RTM $\mathcal{U}$ is *universal* (given some coding of RTMs) if for every RTM $\mathcal{M}$ it holds that $\mathcal{T}(\mathcal{M}) \leftrightarrow_b^\Delta \left[\overline{\mathcal{M}} \parallel \mathcal{U}\right]_{\{u\}}$. However, we will show now that such a universal RTM $\mathcal{U}$ does not exist.

Proposition 3.15. *There is no universal RTM up to divergence-preserving branching bisimulation.*

Proof. Assume the existence of a universal RTM $\mathcal{U}$. Since $\mathcal{U}$ is an RTM, it has an associated transition system that has a branching degree bounded by, say, B. Now, assume an RTM $\mathcal{M}$ such that $\mathcal{T}(\mathcal{M})$ has no divergence and has a branching degree bounded by $B+1$. In particular, $\mathcal{T}(\mathcal{M})$ has a state s that realises the branching degree bound by having transitions $a_1, \ldots, a_{B+1}$ to $B+1$ pairwise non-bisimilar target states. If $\mathcal{U}$ were to simulate $\mathcal{M}$ up to divergence-preserving branching bisimulation, then there is a state s' in $\left[\overline{\mathcal{M}} \parallel \mathcal{U}\right]_{\{u\}}$ related to s that cannot do any (inert) τ-steps, but still has to simulate all transitions of s. This means that s' must have a branching degree of $B+1$. This is a contradiction. □

Therefore, if we insist on having simulation up to divergence-preserving branching bisimulation, then the best possible result is to define a separate universal RTM for each possible branching degree.

Definition 3.16. *An RTM $\mathcal{U}_B$ is* universal up to branching degree B *if for every RTM $\mathcal{M}$ with $\mathcal{T}(\mathcal{M})$ bounded by branching degree B it holds that $\mathcal{T}(\mathcal{M}) \leftrightarrow_b^\Delta \left[\overline{\mathcal{M}} \parallel \mathcal{U}_B\right]_{\{u\}}$.*

We now present the construction of a collection of RTMs $\mathcal{U}_B$ for all branching degree bounds B.

For the remainder of this section let $\mathcal{M} = (\mathcal{S}_M, \rightarrow_M, \uparrow_M, \downarrow_M)$ be an RTM such that the branching degree of $\mathcal{T}(\mathcal{M})$ is bounded by B. From our Definition 2.5, Proposition 3.2, the explanations in [19], and by applying some standard recursion-theoretic techniques such as the enumeration theorem (see [21]), it can be shown that the codes of the functions $out(\_)$ and $fin(\_)$ belonging to $\mathcal{T}(\mathcal{M})$ are recursively computable from $\ulcorner\mathcal{M}\urcorner$. Therefore, we can reuse the simulator RTM presented before; it suffices to adapt its initialisation fragment.

Universal initialisation fragment. Instead of writing the codes of the functions $out(\_)$ and $fin(\_)$ and the initial state directly on the tape, the *universal initialisation fragment* InitU first receives the code $\ulcorner\mathcal{M}\urcorner$ of an arbitrary $\mathcal{M}$ along some dedicated channel u, yielding the tape instance $\ulcorner\mathcal{M}\urcorner$. Let $\mathcal{M}_{ri}$ be an RTM that handles the receiving and storing of the code $\ulcorner\mathcal{M}\urcorner$ over channel u when started from an empty tape.

Then, it recursively computes, from $\ulcorner\mathcal{M}\urcorner$, the codes of the functions $out(\_)$ and $fin(\_)$, and the initial state $\uparrow_\mathcal{M}$ of $\mathcal{T}(\mathcal{M})$ and stores these on the tape. As mentioned before, these functions can be computed recursively, and let $\mathcal{M}_{ci}$ be the deterministic Turing machine that, when started from the tape instance $\ulcorner\mathcal{M}\urcorner$ halts with the tape instance $\llbracket\ulcorner out\urcorner|\ulcorner fin\urcorner\check{\rrbracket}\ulcorner\uparrow_\mathcal{M}\urcorner$.

The set of states of InitU is defined as

$$\mathcal{S}_{\mathsf{InitU}} = (\mathcal{S}_{\mathcal{M}_{ri}} \cup \mathcal{S}_{\mathcal{M}_{ci}}) \setminus (\downarrow_{\mathcal{M}_{ri}} \cup \downarrow_{\mathcal{M}_{ci}}) \ ,$$

its initial state is defined as

$$\uparrow_{\mathsf{InitU}} = \uparrow_{\mathcal{M}_{ri}} \ ; \text{ and}$$

its set of transitions is defined as

$$\begin{aligned}\rightarrow_{\mathsf{InitU}} = &\{(in, d, \tau, e, M, in') \mid (in, d, \tau, e, M, in') \in \rightarrow_{\mathcal{M}_{ri}},\ in' \in \mathcal{S}_{\mathcal{M}_{ri}} \setminus \downarrow_{\mathcal{M}_{ri}}\}\\ &\cup \{(in, d, \tau, e, M, \uparrow_{\mathcal{M}_{ci}}) \mid (in, d, \tau, e, M, in') \in \rightarrow_{\mathcal{M}_{ri}},\ in' \in \downarrow_{\mathcal{M}_{ri}}\}\\ &\cup \{(in, d, \tau, e, M, in') \mid (in, d, e, M, in') \in \rightarrow_{\mathcal{M}_{ci}},\ in' \in \mathcal{S}_{\mathcal{M}_{ci}} \setminus \downarrow_{\mathcal{M}_{ci}}\}\\ &\cup \{(in, d, \tau, e, M, \uparrow_{\mathsf{State}}) \mid (in, d, e, M, in') \in \rightarrow_{\mathcal{M}_{ci}},\ in' \in \downarrow_{\mathcal{M}_{ci}}\}\end{aligned}$$

Note that Lemma 3.6 holds for this fragment InitU as well, albeit that the path constitutes of a different set of configurations.

Lemma 3.17. *The fragment* InitU *has a deterministic computation from* $(\uparrow_{\mathsf{InitU}}, \check{\Box})$ *to* $(\uparrow_{\mathsf{State}}, [\![\ulcorner out \urcorner | \ulcorner fin \urcorner]\!\check{]}\ulcorner \uparrow_{\mathcal{M}} \urcorner)$.

Universal RTMs. Now, when the universal initialisation fragment sets up the simulation, the state and step fragments, that have already been defined in the previous section, can perform the simulation as before. We define the *universal RTM* $\mathcal{U}_B = (\mathcal{S}_{\mathcal{U}_B}, \rightarrow_{\mathcal{U}_B}, \uparrow_{\mathcal{U}_B}, \downarrow_{\mathcal{U}_B})$ for each branching degree B as the combination of the fragments InitU, State and Step defined above. Recall that the fragment Step contains states for every possible menu but that these menus have a branching degree that is bounded by B. Because of this we can reuse the step fragment; the definition of fragment is independent of the transition function it is simulating and only parametrised by the branching degree bound B.

The set of states of each particular $\mathcal{U}_B$ is defined as the union of the sets of states of the fragments:

$$\mathcal{S}_{\mathcal{U}_B} = \mathcal{S}_{\mathsf{InitU}} \cup \mathcal{S}_{\mathsf{State}} \cup \mathcal{S}_{\mathsf{Step}} \ \ ;$$

the transition relation of $\mathcal{U}_B$ is the union of the transition relations of all fragments:

$$\mathcal{S}_{\mathcal{U}_B} = \rightarrow_{\mathsf{InitU}} \cup \rightarrow_{\mathsf{State}} \cup \rightarrow_{\mathsf{Step}} \ \ ;$$

the initial state of $\mathcal{U}_B$ is the initial state of InitU:

$$\uparrow_{\mathcal{U}_B} = \uparrow_{\mathsf{InitU}} \ \ ; \text{ and}$$

the set of final states of $\mathcal{U}_B$ consists of the states of Step $sp_{(s \in \downarrow_T)?, a_1, \ldots, a_k}$ where s is a final configuration in $\mathcal{T}(\mathcal{M})$

$$\downarrow_{\mathcal{U}_B} = \{sp_{(s \in \downarrow_T)?, a_1, \ldots, a_k} \mid s \in \downarrow_T\} \ \ .$$

Theorem 3.18. *For every B there exists an RTM $\mathcal{U}_B$ such that, for all RTMs $\mathcal{M}$ with a branching degree bounded by B, it holds that* $\mathcal{T}(\mathcal{M}) \leftrightarrow_b^{\Delta} \left[\overline{\mathcal{M}} \parallel \mathcal{U}_B\right]_{\{u\}}$.

If we drop the requirement that the simulation has to be divergence-preserving, we can find a single universal RTM. We replace the Turing machine $\mathcal{M}_{ci}$ in the fragment InitU by an adapted version that besides calculating $out(\_)$ and $fin(\_)$ also modifies $out(\_)$ to reduce the branching degree to at most 2 [2]. This is, necessarily, at the cost of introducing divergence. The resulting universal RTM $\mathcal{U}'$ is universal up to branching bisimulation.

Theorem 3.19. *There exists an RTM $\mathcal{U}'$ such that, for all RTMs $\mathcal{M}$, it holds that* $\mathcal{T}(\mathcal{M}) \leftrightarrow_b \left[\overline{\mathcal{M}} \parallel \mathcal{U}'\right]_{\{u\}}$.

4 Explicit Interaction

In this section we show that, up to divergence-preserving branching bisimilarity, every executable transition system can be specified using the process theory TCP_τ [1]. We do this by showing, for any given RTM, the construction of a *finite* recursive specification over TCP_τ that simulates its behaviour. Our specification will consist of a finite specification of a process that is a translated version of the finite control of the RTM, and a finite specification of tape memory. We shall prove that the parallel composition of these specifications specifies a transition system that is divergence-preserving branching bisimilar with the transition associated with the RTM. Further note that our specification deals explicitly with the interaction between the finite control and the tape of an RTM. This result follows up on results in earlier work [5, 6] where we have shown in similar vain that the interaction within pushdown automata and bag automata can be made explicit.

It follows from results obtained by Vaandrager in [25] that every TCP_τ-specification induces an effective transition system. Hence, by Corollary 3.11, we also get the converse: every transition system definable in TCP_τ is executable up to branching bisimilarity.

Since transition systems associated with TCP_τ-specifications can be simulated, up to branching bisimulation, by a finite control interacting with a queue (we will later see that the tape process wraps a queue with some finite control), we can look upon the queue as the prototypical TCP_τ process.

We could argue that TCP_τ-specifications can be considered as the process-theoretic counterparts of unrestricted grammars. In automata and formal language theory a hierarchy of classes of languages with different expressivity is obtained by adding/dropping restrictions on the left-hand and right-hand side of grammars. In process theory, the stricter recursive specification format is used, and different classes of expressivity are obtained by allowing more/less operators, notably the parallel composition here, in the right-hand sides. This we have also shown for regular expressions in [7].

4.1 TCP_τ

First, we introduce an instance of TCP_τ, for the full definition see [1], with the specific form of handshaking communication from [4]. TCP_τ is a generic process algebra encompassing key features of ACP, CCS and CSP.

Recall that we reuse the finite set $\mathcal{C}$ of channels and set of data $\mathcal{D}_\Box$; we introduce the set of special actions $\mathcal{I} = \{c?d, c!d, c?\!\!!d \mid d \in \mathcal{D}_\Box, c \in \mathcal{C}\}$. The actions $c?d$, $c!d$, $c?\!\!!d$ respectively denote the event that a datum d is received, sent, or communicated along *channel* c. Let $\mathcal{N}$ be a countably infinite set of names. The set of *process expressions* $\mathcal{P}$ is generated by the following grammar ($a \in \mathcal{A}_\tau \cup \mathcal{I}, N \in \mathcal{N}, c \in \mathcal{C}$):

$$p ::= \mathbf{0} \mid \mathbf{1} \mid a.p \mid p \cdot p \mid p + p \mid p \parallel p \mid p \mathbin{\|\!\!\!\lfloor} p \mid p \mid p \mid \partial_c(p) \mid \tau_c(p) \mid N \,.$$

Let us briefly comment on the operators in this syntax. The constant $\mathbf{0}$ denotes *deadlock*, the unsuccessfully terminated process. The constant $\mathbf{1}$ denotes *skip*, the successfully terminated process. For each action $a \in \mathcal{A} \cup \mathcal{I}$ there is a unary operator $a.$ denoting action prefix; the process denoted by $a.p$ can do an a-transition to the process denoted by p. The binary operator $\cdot$ denotes *sequential composition*. The binary operator $+$ denotes *alternative composition* or *choice*. The binary operator $\parallel$ denotes *parallel composition*; actions of both arguments are interleaved, and in addition a communication $c?\!\!!d$ of a datum d on channel c can take place if one argument can do an input action $c?d$ that matches an output action $c!d$ of the other component. The left-merge $\mathbin{\|\!\!\!\lfloor}$ and communication merge $\mid$ are auxiliary operators needed for the axiomatisation that we will see later on. The unary operator $\partial_c(p)$

encapsulates the process p in such a way that all input actions $c?d$ and output actions $c!d$ are blocked (for all data) so that communication is enforced. Finally, the unary operator $\tau_c(p)$ denotes abstraction from communication over channel c in p by renaming all communications $c\,\text{!`}d$ to τ-transitions. We shall abbreviate $\tau_c(\partial_c(p))$ with $[p]_c$.

A *recursive specification* E is a set of equations of the form: $N \stackrel{\text{def}}{=} p$, with as left-hand side a name N and as right-hand side a process expression p. It is required that a recursive specification E contains, for every $N \in \mathcal{N}$, at most one equation with N as left-hand side; this equation will be referred to as the *defining equation* for N in $\mathcal{N}$. Furthermore, if some name occurs in the right-hand side of some defining equation, then the recursive specification must include a defining equation for it.

We use Structural Operational Semantics [20] to associate a transition relation with process expressions: let $\rightarrow$ be the $\mathcal{A}_\tau$-labelled transition relation induced on the set of process expressions by operational rules in Table 1. Note that the operational rules presuppose a recursive specification E.

Definition 4.1. *Let E be a recursive specification and let p be a process expression. We define the labelled transition system $\mathcal{T}_E(p) = (\mathcal{S}_p, \rightarrow_p, \uparrow_p, \downarrow_p)$ associated with p and E as follows:*

1. *the set of states $\mathcal{S}_p$ consists of all process expressions reachable from p;*
2. *the transition relation $\rightarrow_p$ is the restriction to $\mathcal{S}_p$ of the transition relation $\rightarrow$ defined on all process expressions by the operational rules in Table 1, i.e., $\rightarrow_p = \rightarrow \cap (\mathcal{S}_p \times \mathcal{A}_\tau \times \mathcal{S}_p)$.*
3. *the process expression p is the initial state, i.e. $\uparrow_p = p$; and*
4. *the set of final states consists of all process expressions $q \in \mathcal{S}_p$ such that $q\downarrow$, i.e., $\downarrow_p = \downarrow \cap \mathcal{S}_p$.*

To be able to give concise proofs that certain process expressions are divergence-preserving branching bisimilar, it is convenient to proceed by equational reasoning. We shall use the equations in Table 2, see [1] for an explanation of the axioms, and the proof rule RSP [1], which is based on the assumption that every guarded recursive specification has a unique solution. (Actually, the guardedness of the specifications below follows from the fact that they are τ-guarded and τ-founded, as defined in [9].)

We should, of course, establish that an equational reasoning based on the axioms in Table 2 is sound, i.e., that it indeed proves that the equated process expressions are divergence-preserving branching bisimilar. For this it suffices to prove that the axioms in Table 2 and RSP are sound with respect to some congruence included in divergence-preserving branching bisimilarity. (Note that divergence-preserving branching bisimilarity is not a congruence with respect to the operator $+$ for the same reason as why branching bisimilarity is not a congruence with respect to $+$). The way we obtain a congruence included in divergence-preserving branching bisimilarity is standard: we define a *rooted* version:

Definition 4.2. *A divergence-preserving branching bisimulation $\mathcal{R}$ from T_1 to T_2 is called* rooted *if it meets the following root-conditions:*

1. *for all states $s_1' \in \mathcal{S}_1$, whenever $\uparrow_1 \xrightarrow{a} s_1'$, then there exists a state s_2' such that $\uparrow_2 \xrightarrow{a} s_2'$ and $s_1' \mathrel{\mathcal{R}} s_2'$;*
2. *for all states $s_2' \in \mathcal{S}_2$, whenever $\uparrow_2 \xrightarrow{a} s_2'$, then there exists a state s_1' such that $\uparrow_1 \xrightarrow{a} s_1'$ and $s_1' \mathrel{\mathcal{R}} s_2'$;*

$$\frac{}{\mathbf{1}\downarrow} \qquad \frac{}{a.p \xrightarrow{a} p}$$

$$\frac{p \xrightarrow{a} p'}{p + q \xrightarrow{a} p'} \qquad \frac{q \xrightarrow{a} q'}{p + q \xrightarrow{a} q'} \qquad \frac{p\downarrow}{(p+q)\downarrow} \qquad \frac{q\downarrow}{(p+q)\downarrow}$$

$$\frac{p \xrightarrow{a} p'}{p \cdot q \xrightarrow{a} p' \cdot q} \qquad \frac{p\downarrow \quad q \xrightarrow{a} q'}{p \cdot q \xrightarrow{a} q'} \qquad \frac{p\downarrow \quad q\downarrow}{(p \cdot q)\downarrow}$$

$$\frac{p \xrightarrow{a} p'}{p \parallel q \xrightarrow{a} p' \parallel q} \qquad \frac{q \xrightarrow{a} q'}{p \parallel q \xrightarrow{a} p \parallel q'} \qquad \frac{p\downarrow \quad q\downarrow}{(p \parallel q)\downarrow}$$

$$\frac{p \xrightarrow{c!d} p' \quad q \xrightarrow{c?d} q'}{p \parallel q \xrightarrow{c?!d} p' \parallel q'} \qquad \frac{p \xrightarrow{c?d} p' \quad q \xrightarrow{c!d} q'}{p \parallel q \xrightarrow{c?!d} p' \parallel q'}$$

$$\frac{p \xrightarrow{a} p' \quad a \neq c?d, c!d}{\partial_c(p) \xrightarrow{a} \partial_c(p')} \qquad \frac{p\downarrow}{\partial_c(p)\downarrow}$$

$$\frac{p \xrightarrow{c?!d} p'}{\tau_c(p) \xrightarrow{\tau} \tau_c(p')} \qquad \frac{p \xrightarrow{a} p' \quad a \neq c?!d}{\tau_c(p) \xrightarrow{a} \tau_c(p')} \qquad \frac{p\downarrow}{\tau_c(p)\downarrow}$$

$$\frac{p \xrightarrow{a} p' \quad (N \stackrel{\text{def}}{=} p) \in E}{N \xrightarrow{a} p'} \qquad \frac{p\downarrow \quad (N \stackrel{\text{def}}{=} p) \in E}{N\downarrow}$$

Table 1: Operational rules for a recursive specification E.

A1	$x + y = y + x$	A6	$x + \mathbf{0} = x$
A2	$(x + y) + z = x + (y + z)$	A7	$\mathbf{0} \cdot x = \mathbf{0}$
A3	$x + x = x$	A8	$x \cdot \mathbf{1} = x$
A4	$(x + y) \cdot z = x \cdot z + y \cdot z$	A9	$\mathbf{1} \cdot x = x$
A5	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	A10	$a.x \cdot y = a.(x \cdot y)$
M	$x \parallel y = x \mathbin{⌊\!⌊} y + y \mathbin{⌊\!⌊} x + x \mid y$	B	$a.(\tau.(x + y) + x) = a.(x + y)$
LM1	$\mathbf{0} \mathbin{⌊\!⌊} x = \mathbf{0}$	SC1	$x \mid y = y \mid x$
LM2	$\mathbf{1} \mathbin{⌊\!⌊} x = \mathbf{0}$	SC2	$x \parallel \mathbf{1} = x$
LM3	$a.x \mathbin{⌊\!⌊} y = a.(x \parallel y)$	SC3	$\mathbf{1} \mid x + \mathbf{1} = \mathbf{1}$
LM4	$(x + y) \mathbin{⌊\!⌊} z = x \mathbin{⌊\!⌊} z + y \mathbin{⌊\!⌊} z$	SC4	$(x \parallel y) \parallel z = x \parallel (y \parallel z)$
CM1	$\mathbf{0} \mid x = \mathbf{0}$	SC5	$(x \mid y) \mid z = x \mid (y \mid z)$
CM2	$(x + y) \mid z = x \mid z + y \mid z$	SC6	$(x \mathbin{⌊\!⌊} y) \mathbin{⌊\!⌊} z = x \mathbin{⌊\!⌊} (y \parallel z)$
CM3	$\mathbf{1} \mid \mathbf{1} = \mathbf{1}$	SC7	$(x \mid y) \mathbin{⌊\!⌊} z = x \mid (y \mathbin{⌊\!⌊} z)$
CM4	$a.x \mid \mathbf{1} = \mathbf{0}$	SC8	$x \mathbin{⌊\!⌊} \mathbf{0} = x \cdot \mathbf{0}$
CM5	$c!d.x \mid c?d.y = c?!d.(x \parallel y)$	SC9	$x \mathbin{⌊\!⌊} \tau.y = x \mathbin{⌊\!⌊} y$
CM6	$a.x \mid b.y = \mathbf{0}$ if $\{a, b\} \neq \{c!d, c?d\}$	SC10	$x \mid \tau.y = \mathbf{0}$
D1	$\partial_c(\mathbf{1}) = \mathbf{1}$	T1	$\tau_c(\mathbf{1}) = \mathbf{1}$
D2	$\partial_c(\mathbf{0}) = \mathbf{0}$	T2	$\tau_c(\mathbf{0}) = \mathbf{0}$
D3	$\partial_c(a.x) = \mathbf{0}$ if $a = c?d, c!d$	T3	$\tau_c(a.x) = a.\tau_c(x)$ if $a \neq c?d, c!d$
D4	$\partial_c(a.x) = a.\partial_c(x)$ if $a \neq c?d, c!d$	T4	$\tau_c(a.x) = \tau.\tau_c(x)$ if $a = c?d, c!d$
D5	$\partial_c(x + y) = \partial_c(x) + \partial_c(y)$	T5	$\tau_c(x + y) = \tau_c(x) + \tau_c(y)$

Table 2: The axioms of the process theory TCP_τ ($a \in \mathcal{A}, d \in \mathcal{D}_\Box$).

3. *if* $\uparrow_1\downarrow_1$, *then* $\uparrow_2\downarrow_2$;

4. *if* $\uparrow_2\downarrow_2$, *then* $\uparrow_1\downarrow_1$.

The transition systems T_1 *and* T_2 *are* divergence-preserving rooted branching bisimilar *(notation: $T_1 \leftrightarrow^{\Delta}_{rb} T_2$) if there exists a divergence-preserving branching bisimulation from T_1 to T_2 that meets the above mentioned root-conditions.*

Since divergence-preserving branching bisimilarity is included in rooted divergence-preserving branching bisimilarity, we have the following proposition that we will use in the proofs below.

Proposition 4.3. *The equational theory given by Table 2 is sound for the model of transition systems modulo divergence-preserving branching bisimilarity.*

Note that the KFAR axiom [2] is not a part of the axioms because that could lead to the removal of τ-loops which would break the divergence-preserving property.

4.2 Correspondence

We prove that for every reactive Turing machine $\mathcal{M}$ there exists a finite recursive TCP$_\tau$-specification $E_{\mathcal{M}}$ and process expression p such that $\mathcal{T}(\mathcal{M}) \leftrightarrow^{\Delta}_{b} \mathcal{T}_{E_{\mathcal{M}}}(p)$. For clarity purposes we will present $E_{\mathcal{M}}$ in two steps. First, we will consider a finite recursive specification a tape process E_T and show its correspondence with an infinite specification of the tape process. Then, we will present a fair translation of the finite control of an RTM into a finite recursive specification E_{fc}. We conclude by showing that the correspondence of the combined finite specification $E_{\mathcal{M}}$ with the original RTM $\mathcal{M}$ holds.

The tape. The following infinite recursive specification E_T^∞ specifies the desired behaviour and interface of a tape process $T_{\delta_L \check{d} \delta_R}$ for every possible tape instance ($d \in \mathcal{D}_\Box, \delta_L, \delta_R \in \mathcal{D}_\Box^*$). Each name has an equation that expresses that the datum d under the head can be sent over channel r (read), a datum e can be received over channel w (write) to replace the datum under the head, and commands can be received over channel m (move) to move the head one position to the left (onto δ_L) or right (onto δ_R); each name has the following defining equation:

$$T_{\delta_L \check{d} \delta_R} \stackrel{\text{def}}{=} \mathbf{1} + r!d.T_{\delta_L \check{d} \delta_R} + \sum_{e \in \mathcal{D}_\Box^*} w?e.T_{\delta_L \check{e} \delta_R} + m?L.T_{\delta_L{}^{<} d \delta_R} + m?R.T_{\delta_L d\,{}^{>}\delta_R} .$$

Notice that this specification allows reading and writing and moving independently, as it was also originally defined by Turing in [23].

The specification above of the tape process is clearly infinite, since we have a name for each possible tape instance. Our aim is, however, to have a finite specification. In earlier work by Baeten, Bergstra and Klop in [2] a finite specification of a Turing machine is given in ACP$_\tau$ to simulate computable transition systems up to bisimilarity; the conventional Turing machine is simulated using finite control in parallel with two stacks. Their approach to model a tape as two stacks cannot be reused in our settings, which allows for states that can be terminating and have outgoing transitions at the same time. Their specification of the stack does not allow for intermediate termination, and it is not clear how to adapt it, so that it does. Instead, we model the tape using a (first-in first-out) queue, which does allow for intermediate termination.

The following infinite linear recursive specification E_Q^∞ specifies the behaviour of the process Q_δ modelling a queue with contents δ that receives input over channel i and sends output over channel o (for every $d \in \mathcal{D}_\Box, \delta \in \mathcal{D}_\Box^*$):

$$Q_\varepsilon \stackrel{\text{def}}{=} \mathbf{1} + \sum_{d \in \mathcal{D}} i?d.Q_d \ , \qquad Q_{\delta d} \stackrel{\text{def}}{=} \mathbf{1} + o!d.Q_\delta + \sum_{e \in \mathcal{D}} i?e.Q_{e\delta d} \ .$$

Since we want the queue process to have a finite specification too, we use as a basis for the finite version the recursive specification originally given by Bergstra and Klop in [10], which uses six variables, parallel composition, communication over an input, output and auxiliary channel and abstraction. Bezem and Ponse have shown in [11] that this finite recursive specification is branching bisimilar (without the terminations conditions 3 and 4 of Definition 2.8) with the infinite recursive specification given above. In their proof, they also show that the finite recursive specification does not have infinite τ-paths, so in effect they show divergence-preserving branching bisimilarity. The following specification E_Q is an adaptation of the finite specification of Bergstra and Klop defining a version of the queue that always has the option to terminate.

$$Q_k^{i,j} \stackrel{\text{def}}{=} \mathbf{1} + \sum_{d \in \mathcal{D}_\Box} i?d.\left[Q_j^{i,k} \parallel (\mathbf{1} + j!d.Q_i^{k,j})\right]_k \quad \text{for all } \{i,j,k\} = \{i,o,l\}.$$

The adaptation with respect to Bergstra and Klop's specification consists of the addition of a **1**-summand to the defining equation of every variable and to right-most component of the therein contained parallel composition. As a result, termination can occur in every state of the queue, and no other change in behaviour is incurred. Thus, similarly to [11] it can be proved that our infinite recursive specification is divergence-preserving branching bisimilar—this time with the termination conditions—with the finite recursive specification given above.

Lemma 4.4. *We have that* $Q_\varepsilon \leftrightarrow_b^\Delta Q_l^{io}$.

This lemma also allows us to use the more concise notation of the infinite specification, Q_δ for some $\delta \in \mathcal{D}_\Box{}^*$, for a state of the queue process given by the finite specification in the proofs below.

We can now define the finite recursive specification of the tape process E_T as the finite recursive specification of the queue E_Q and the following equations ($d \in \mathcal{D}_\Box$):

$$H_d \stackrel{\text{def}}{=} \mathbf{1} + r!d.H_d + \sum_{e \in \mathcal{D}_\Box} w?e.H_e + m?L.H_d^L + m?R.H_d^R\ ,$$

$$H_d^L \stackrel{\text{def}}{=} i!d.\Big(\sum_{e \in \mathcal{D}_\Box} o?e.H_e + o?\bot.i!\$.i!\bot.Back\Big)\ ,$$

$$Back \stackrel{\text{def}}{=} \sum_{d \in \mathcal{D}_\Box} o?d.i!d.Back + o?\$.H_\Box\ ,$$

$$H_d^R \stackrel{\text{def}}{=} i!\$.i!d.\Big(\sum_{e \in \mathcal{D}_\Box} o?e.Fwd_e + o?\bot.Fwd_\bot\Big)\ ,$$

$$Fwd_d \stackrel{\text{def}}{=} \sum_{e \in \mathcal{D}_\Box} o?e.i!d.Fwd_e + o?\bot.i!d.Fwd_\bot + o?\$.H_d\ ,$$

$$Fwd_\bot \stackrel{\text{def}}{=} \sum_{e \in \mathcal{D}_\Box} o?e.i!\bot.Fwd_e + o?\$.i!\bot.H_\Box\ .$$

Unlike the stack, the queue allows us to reach any arbitrary data element contained within in a non-destructive way. We can repeatedly remove a datum from the queue over channel o and then insert it over channel i; we call this *shifting*. Doing this once is called a *shift operation*. Although shifting suggests the usage of a queue in a circular fashion, we have to map the (infinite and linear) data structure of the tape onto the queue. We use the queue to store only the part of the tape to the left of the head δ_L and to the right of the head δ_R and we keep the datum under the head d in a separate head process H_d. Additionally we use the marker $\bot$ as special queue data element to separate the left from the right part and also

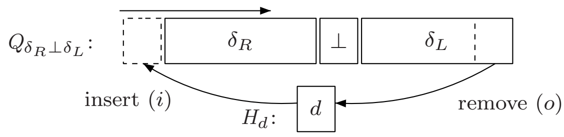

Figure 6: Diagram of the tape process.

to indicate that the tape can be extended on the left or on the right, when needed, by inserting elements between $\bot$ and δ_L or between δ_R and $\bot$ respectively. Fig. 6 illustrates the mapping of the tape instance $\delta_L\check{d}\delta_R$ and a shift operation.

In the recursive specification E_T above the main process H_d models the situation that the datum d is at the position of the head. This process H_d is put in parallel with the queue process $Q_{\delta_R\bot\delta_L}$ and provides the interface to the tape. Read and write operations for the tape are dealt with by the head process without accessing the queue; shifting only occurs when a move is requested. This is another reason to have a separate head process that directly handles a read and write operation without touching the queue: if the datum at the position of the head would be on the queue as well, every read or write operation for the tape would cause shifting and require additional operations to get the queue in the right state again.

As mentioned above, moving the head to the left—handled by H_d^L—requires just one shift operation. However, we have to make sure not to remove the special marker $\bot$ after inserting datum d in the case that the sequence to the left of the head (δ_L) is empty. If this happens, we insert a search marker \$ followed by $\bot$ and cycle through the queue completely until \$ reappears. We get the following lemma that establishes that a move to the left behaves as expected using a fixed number of internal steps.

Lemma 4.5. *For every $d \in \mathcal{D}_\Box, \delta_L, \delta_R \in \mathcal{D}_\Box{}^*$ we have that*

$$\left[H_d^L \parallel Q_{\delta_R\bot\delta_L}\right]_{io} \mathrel{\underline{\leftrightarrow}}_b^\Delta \tau. \begin{cases} [H_{d_L} \parallel Q_{d\delta_R\bot\zeta_L}]_{io} & \text{if } \delta_L = \zeta_L d_L\ , \\ [H_\Box \parallel Q_{d\delta_R\bot}]_{io} & \text{if } \delta_L = \varepsilon\ . \end{cases} \tag{1}$$

Proof. We prove the validity of equation 1 by means of an equational reasoning using the axioms of Table 2 and RSP. Then, the lemma follows by Proposition 4.3. We distinguish two cases for δ_L in $\left[H_d^L \parallel Q_{\delta_R\bot\delta_L}\right]_{io}$:

1. If $\delta_L = \zeta_L d_L$, then H_d^L moves the tape head to the left by performing one shift operation. So, first the datum under the head d is prefixed to the sequence to right of the head (δ_R), and then the right-most datum (d_L) of the sequence to the left of the head (δ_L) is removed and put it under the head (see also Figure 6).

$$\left[H_d^L \parallel Q_{\delta_R\bot\zeta_L d_L}\right]_{io} = \tau.\tau.\left[H_{d_L}^L \parallel Q_{d\delta_R\bot\zeta_L}\right]_{io} = \tau.\left[H_{d_L}^L \parallel Q_{d\delta_R\bot\zeta_L}\right]_{io}\ .$$

2. If $\delta_L = \varepsilon$, then H_d^L initially removes the special $\bot$ symbol from the queue, inserts the special search marker \$, reinserts $\bot$ and then switches to *Back*. This will shift through the queue contents until \$ is reached. At this point the queue is consistent again, so it removes the search marker and the blank symbol is put under the head.

$$\begin{aligned} \left[H_d^L \parallel Q_{\delta_R\bot}\right]_{io} &= \tau.\tau.\tau.\tau.\left[Back \parallel Q_{\bot\$d\delta_R}\right]_{io} \\ &= \tau.\tau.\tau.\tau.\tau^{2|d\delta_R|}.\left[Back \parallel Q_{d\delta_R\bot\$}\right]_{io} \\ &= \tau.\tau.\tau.\tau.\tau^{2|d\delta_R|}.\tau.\left[H_\Box \parallel Q_{d\delta_R\bot}\right]_{io} \\ &= \tau.\left[H_\Box \parallel Q_{d\delta_R\bot}\right]_{io}\ . \end{aligned}$$

We can observe that there is a fixed upper bound of $2|d\delta_R| + 5$ to the number of τ-steps (in the second case). Hence, there is no divergence. □ □

Because shifting through the queue contents only goes in one direction, we have to use a different approach for moving the head to the right, which is handled by H_d^R. This time we need to have the left-most datum of the sequence to the right of the queue (δ_R) and we will have to shift through the entire queue contents to reach it. We do this by inserting a search marker \$ into the queue and shifting through it using a lookahead that remembers the datum that was previously removed from the queue. Once we encounter the search marker, we put this previously encountered datum under the head.

Lemma 4.6. *For every $d \in \mathcal{D}_\Box, \delta_L, \delta_R \in {\mathcal{D}_\Box}^*$ we have that*

$$\left[H_d^R \parallel Q_{\delta_R \perp \delta_L}\right]_{io} \leftrightarrow_b^\Delta \tau.\begin{cases}[H_{d_R} \parallel Q_{\zeta_R \perp \delta_L d}]_{io} & \text{if } \delta_R = d_R\zeta_R\,,\\ [H_\Box \parallel Q_{\perp \delta_L d}]_{io} & \text{if } \delta_R = \varepsilon\,.\end{cases} \tag{2}$$

Proof. We prove the validity of equation 1 by means of an equational reasoning using the axioms of Table 2 and RSP. Then, the lemma follows by Proposition 4.3.

$$\begin{aligned}\left[H_d^R \parallel Q_{\delta_R \perp \delta_L}\right]_{io} &= \tau.\tau.\tau^{2|\delta_L|+1}.\,[Fwd_\perp \parallel Q_{\delta_L d \$ \delta_R}]_{io}\\ &= \begin{cases}\tau.\tau.\tau^{2|\delta_L|+1}.\tau^{2|d_R\delta_R|}.\tau.\,[H_{d_R} \parallel Q_{\delta_R \perp \delta_L d}]_{io} & \text{if } \delta_R = d_R\zeta_R\\ \tau.\tau.\tau^{2|\delta_L|+1}.\tau.\tau.\,[H_\Box \parallel Q_{\perp \delta_L d}]_{io} & \text{if } \delta_R = \varepsilon\end{cases}\\ &= \tau.\begin{cases}[H_{d_R} \parallel Q_{\delta_R \perp \delta_L d}]_{io} & \text{if } \delta_R = d_R\zeta_R\\ [H_\Box \parallel Q_{\perp \delta_L d}]_{io} & \text{if } \delta_R = \varepsilon\,.\end{cases}\end{aligned}$$

We can observe that there is a fixed upper bound of $2|\delta_L d_R \delta_R|+4$ to the number of τ-steps. Hence, there is no divergence. □ □

Putting everything together, we get the following result that shows that behavioural specification of the tape E_T^∞ is divergence-preserving branching bisimilar with the finite specification E_T.

Lemma 4.7. *For each tape instance $\delta_L\check{d}\delta_R$ ($\delta_L, \delta_R \in \mathcal{D}_\Box^*, d \in \mathcal{D}_\Box$) we have that $T_{\delta_L\check{d}\delta_R} \leftrightarrow_b^\Delta [H_d \parallel Q_{\delta_R \perp \delta_L}]_{io}$.*

Proof. We prove the validity of equation by means of an equational reasoning using the axioms of Table 2 and RSP. Then, the lemma follows by Proposition 4.3.

$$T_{\delta_L\check{d}\delta_R} = [H_d \parallel Q_{\delta_R \perp \delta_L}]_{io}$$

Now, expand the expression using axiom M and move the initial actions of H_d outside:

$$\begin{aligned}&= \mathbf{1} + r!d.\,[H_d \parallel Q_{\delta_R\perp\delta_L}]_{io} + w?e.\,[H_e \parallel Q_{\delta_R\perp\delta_L}]_{io}\\ &\quad + m?L.\,\left[H_d^L \parallel Q_{\delta_R\perp\delta_L}\right]_{io} + m?R.\,\left[H_d^R \parallel Q_{\delta_R\perp\delta_L}\right]_{io}\end{aligned}$$

By applying Lemma 4.5 and 4.6 and axiom B we get:

$$\begin{aligned}&= \mathbf{1} + r!d.\,[H_d \parallel Q_{\delta_R\perp\delta_L}]_{io} + w?e.\,[H_e \parallel Q_{\delta_R\perp\delta_L}]_{io}\\ &\quad + m?L.\tau.\begin{cases}[H_{d_L} \parallel Q_{d\delta_R\perp\zeta_L}]_{io} & \text{if } \delta_L = \zeta_L d_L\\ [H_\Box \parallel Q_{d\delta_R\perp}]_{io} & \text{if } \delta_L = \varepsilon\end{cases}\\ &\quad + m?R.\tau.\begin{cases}[H_{d_R} \parallel Q_{\zeta_R\perp\delta_L d}]_{io} & \text{if } \delta_R = d_R\zeta_L\\ [H_\Box \parallel Q_{\perp\delta_L d}]_{io} & \text{if } \delta_R = \varepsilon\end{cases}\\ &= \mathbf{1} + r!d.\,T_{\delta_L\check{d}\delta_R} + w?e.\,T_{\delta_L\check{e}\delta_R} + m?L.\,T_{\delta_L{}^<d\delta_R} + m?R.\,T_{\delta_L d^>\delta_R}\,.\end{aligned}$$

We can observe that there are no τ-loops introduced by the specification. When moving left or right either one shift operation happens or we shift until the search marker is found, both yield a finite number of τ-steps. Hence, no divergence is introduced. □ □

Finite control. Let $\mathcal{M} = (\mathcal{S}, \rightarrow, \uparrow, \downarrow)$ be some RTM. We can write its associated transition system $\mathcal{T}(\mathcal{M})$ as a linear recursive specification $E^{\infty}_{\mathcal{M}}$, which is infinite if $\mathcal{T}(\mathcal{M})$ is infinite. Here, *linear* means that only $\mathbf{0}, \mathbf{1}$, action prefix and $+$ are used in the right-hand side (compare with right-linear grammars).

This recursive specification $E^{\infty}_{\mathcal{M}}$ contains a name $S_{s,\delta_L\check{d}\delta_R}$ for each reachable configuration $(s, \delta_L\check{d}\delta_R)$ ($s \in \mathcal{S}, d \in \mathcal{D}_\Box, \delta_L, \delta_R \in \mathcal{D}_\Box{}^*$) from the initial configuration $(\uparrow, \check{\Box})$. Each name $S_{s,\delta_L\check{d}\delta_R}$ is defined by the following equation:

$$S_{s,\delta_L\check{d}\delta_R} \stackrel{\text{def}}{=} \sum_{(s,d,a,e,L,t)\in\rightarrow} a.S_{t,\delta_L{}^<e\delta_R} + \sum_{(s,d,a,e,R,t)\in\rightarrow} a.S_{t,\delta_L e{}^>\delta_R} \;[+\,\mathbf{1}]_{s\downarrow} \,.$$

Here, $[+\,\mathbf{1}]_{s\downarrow}$ indicates that the $\mathbf{1}$-summand is only present if s is a final state. By construction the transition system $\mathcal{T}_{E^{\infty}_{\mathcal{M}}}(S_{\uparrow,\check{\Box}})$ is isomorphic with $\mathcal{T}(\mathcal{M})$.

Proposition 4.8. *The transition system $\mathcal{T}(\mathcal{M})$ is divergence-preserving branching bisimilar with $\mathcal{T}_{E^{\infty}_{\mathcal{M}}}(S_{\uparrow,\check{\Box}})$.*

Now that we have captured the behaviour of an RTM with an infinite recursive specification, it remains to construct a finite recursive specification and show that it is divergence-preserving branching bisimilar. We now present a finite recursive specification E_{fc} for the finite control of $\mathcal{M}$. For every state $s \in \mathcal{S}$ and datum $d \in \mathcal{D}_\Box$ we add the name $C_{s,d}$ to E_{fc} with the following equation ($s, t \in \mathcal{S}$, $a \in \mathcal{A}_\tau$, $d, e \in \mathcal{D}_\Box$, $M \in \{L, R\}$):

$$C_{s,d} \stackrel{\text{def}}{=} \sum_{(s,d,a,e,M,t)\in\rightarrow} \left(a.w!e.m!M. \sum_{f\in\mathcal{D}_\Box} r?f.C_{t,f} \right) [+\,\mathbf{1}]_{s\downarrow} \,.$$

In E_{fc} each name $C_{s,d}$ represents the part of the finite control of the RTM execution process where a transition can be chosen based on the current state and datum under the head. Once some action a is non-deterministically chosen, the tape—as explained above—is instructed over channel w to write datum e on the place under the head, then it is instructed over channel m to move the head to the left or right and finally over channel r to read the datum f under the new position of the head.

Now, if we put the finite control in parallel with the tape, we can obtain the following lemma.

Lemma 4.9. *For each configuration $(s, \delta_L\check{d}\delta_R)$ of a reactive Turing machine $\mathcal{M}$ we have that $S_{s,\delta_L\check{d}\delta_R} \leftrightarrow^{\Delta}_{b} \left[C_{s,d} \parallel T_{\delta_L\check{d}\delta_R}\right]_{rwm}$.*

Proof. In this proof we want to relate each reachable configuration, represented by the name $S_{s,\delta_L\check{d}\delta_R}$, from the initial configuration of some RTM $\mathcal{M}$ to a name $C_{s,d}$ in the finite control specification E_{fc} put in parallel with a tape process with the corresponding contents, while encapsulating and abstracting from communication between the finite control and tape process. For example, if we have an RTM that has the configuration $(s, \delta_L\check{d}\delta_R)$ and has the transition $s \xrightarrow{a[d/e]L} t$ in its transition relation, then the desired relation between a step in (a part of) the transition system associated with the RTM and the transitions in the specification are shown in Fig. 7.

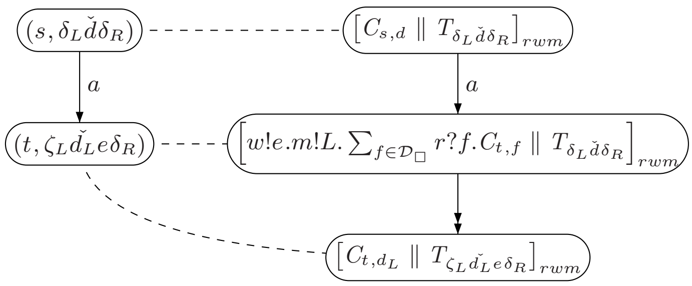

Figure 7: Relation between an RTM transition and specification transitions.

We now proceed to show that $E^{\infty}_{\mathcal{M}}$ is branching bisimilar with $E_{fc} \cup E^{\infty}_{T}$ by means of equational reasoning using the axioms of Table 2 and RSP. Then, the lemma follows by Proposition 4.3.

$$S_{s,\delta_L\check{d}\delta_R} = \left[C_{s,d} \parallel T_{\delta_L\check{d}\delta_R}\right]_{rwm}$$

Unfold $\left[C_{s,d} \parallel T_{\delta_L\check{d}\delta_R}\right]_{rwm}$ and, per transition, move the action outside (by applying almost all of the axioms).

$$= \sum_{(s,d,a,e,M,t)\in\to} a.\left[w!e.m!M.\sum_{f\in\mathcal{D}_\Box} r?f.C_{t,f} \parallel T_{\delta_L\check{d}\delta_R}\right]_{rwm} [+\,\mathbf{1}]_{s\downarrow}$$

Three communications with the tape follow by axiom CM5 and are moved outside by D1–D5 and TI1–TI5.

$$= \sum_{(s,d,a,e,M,t)\in\to} a.\tau.\left[m!M.\sum_{f\in\mathcal{D}_\Box} r?f.C_{t,f} \parallel T_{\delta_L\check{e}\delta_R}\right]_{rwm} [+\,\mathbf{1}]_{s\downarrow}$$

$$= \sum_{(s,d,a,e,L,t)\in\to} a.\tau.\tau.\left[\sum_{f\in\mathcal{D}_\Box} r?f.C_{t,f} \parallel T_{\delta_L{}^{<}e\delta_R}\right]_{rwm} + \sum_{(s,d,a,e,R,t)\in\to} a.\tau.\tau.\left[\sum_{f\in\mathcal{D}_\Box} r?f.C_{t,f} \parallel T_{\delta_L e{}^{>}\delta_R}\right]_{rwm} [+\,\mathbf{1}]_{s\downarrow}$$

$$= \sum_{(s,d,a,e,L,t)\in\to} a.\tau.\tau.\tau.\left[C_{t,f} \parallel T_{\delta_L{}^{<}e\delta_R}\right]_{rwm} + \sum_{(s,d,a,e,R,t)\in\to} a.\tau.\tau.\tau.\left[C_{t,f} \parallel T_{\delta_L e{}^{>}\delta_R}\right]_{rwm} [+\,\mathbf{1}]_{s\downarrow}$$

We can remove the three τ-steps by axiom B.

$$= \sum_{(s,d,a,e,L,t)\in\to} a.\left[C_{t,g} \parallel T_{\delta_L{}^{<}e\delta_R}\right]_{rwm} + \sum_{(s,d,a,e,R,t)\in\to} a.\left[C_{t,g'} \parallel T_{\delta_L e{}^{>}\delta_R}\right]_{rwm} [+\,\mathbf{1}]_{s\downarrow}$$

$$= \sum_{(s,d,a,e,L,t)\in\to} a.S_{t,\delta_L{}^{<}e\delta_R} + \sum_{(s,d,a,e,R,t)\in\to} a.S_{t,\delta_L e{}^{>}\delta_R} \quad [+\,\mathbf{1}]_{s\downarrow}\,.$$

We can observe that no τ-loops or infinite τ-paths are introduced by the specification, nor by the queue as is shown in Lemma 4.5 and 4.6. Hence, there is no divergence. □ □

We have now established a finite version of the specifications for all three components of an RTM. This brings us to the following main result.

Theorem 4.10. *For every reactive Turing machine $\mathcal{M}$ there exists a finite recursive specification $E_{\mathcal{M}}$ and process expression p such that $\mathcal{T}(\mathcal{M}) \leftrightarrow_b^{\Delta} \mathcal{T}_{E_{\mathcal{M}}}(p)$.*

Proof. Choose $E_{\mathcal{M}} = E_{fc} \cup E_T$ and $p = [C_{\uparrow,\Box} \parallel [H_{\Box} \parallel Q_{\perp}]_{io}]_{rwm}$. Then the theorem follows from Property 4.8 and Lemmas 4.4, 4.7, and 4.9. □ □

If we combine the above theorem with Theorem 3.10, Corollary 3.11 and Corollary 3.13 we get the following corollaries.

Corollary 4.11. *Every boundedly branching computable transition system and every deterministic computable transition system is definable, up to to divergence-preserving branching bisimilarity, by a finite* TCP$_{\tau}$*-specification.*

Corollary 4.12. *Every effective transition system is definable, up to branching bisimilarity, by a finite* TCP$_{\tau}$*-specification.*

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References

[1] J. C. M. Baeten, T. Basten, and M. A. Reniers. *Process Algebra (Equational Theories of Communicating Processes)*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2009.

[2] J. C. M. Baeten, J. A. Bergstra, and J. W. Klop. On the consistency of Koomen's fair abstraction rule. *Theoretical Computer Science*, 51:129–176, 1987.

[3] J. C. M. Baeten, P. J. L. Cuijpers, B. Luttik, and P. J. A. van Tilburg. A process-theoretic look at automata. In *Proceedings of FSEN 2009*, number 5961 in LNCS, pages 1–33, Berlin-Heidelberg, 2009. Springer-Verlag.

[4] J. C. M. Baeten, P. J. L. Cuijpers, B. Luttik, and P. J. A. van Tilburg. A process-theoretic look at automata. In F. Arbab and M. Sirjani, editors, *Proceedings of FSEN 2009*, number 5961 in LNCS, pages 1–33, Berlin Heidelberg, 2010. Springer-Verlag.

[5] J. C. M. Baeten, P. J. L. Cuijpers, and P. J. A. van Tilburg. A context-free process as a pushdown automaton. In F. van Breugel and M. Chechik, editors, *Proceedings CONCUR'08*, number 5201 in Lecture Notes in Computer Science, pages 98–113, 2008.

[6] J. C. M. Baeten, P. J. L. Cuijpers, and P. J. A. van Tilburg. A basic parallel process as a parallel pushdown automaton. In D. Gorla and T. Hildebrandt, editors, *Proceedings EXPRESS'08*, Electronic Notes in Theoretical Computer Science, 2009.

[7] J. C. M. Baeten, B. Luttik, T. Muller, and P. J. A. van Tilburg. Expressiveness modulo bisimilarity of regular with parallel composition (extended abstract). In S. B. Fröschle and F. D. Valencia, editors, *Proceedings of EXPRESS 2010*, volume 41 of *EPTCS*, pages 1–15. Open Publishing Association, 2010.

[8] Jos C. M. Baeten, Bas Luttik, and Paul van Tilburg. Computations and interaction. In Raja Natarajan and Adegboyega K. Ojo, editors, *ICDCIT*, volume 6536 of *Lecture Notes in Computer Science*, pages 35–54. Springer, 2011.

[9] J. A. Bergstra, I. Bethke, and A. Ponse. Process algebra with iteration and nesting. *The Computer Journal*, 37(4):243–258, 1994.

[10] J. A. Bergstra and J. W. Klop. Process algebra: Specification and verification in bisimulation semantics. In M. Hazewinkel, J.K. Lenstra, and L. G. L. T. Meertens, editors, *Mathematics and Computer Science II*, volume 4, pages 61–94. North-Holland, Amsterdam, 1986.

[11] M Bezem and A Ponse. Two finite specifications of a queue. *Theoretical Computer Science*, 177(2):487–507, 1997.

[12] A. Blass, Y. Gurevich, D. Rosenzweig, and B. Rossman. Interactive small-step algorithms I: Axiomatization. *Logical Methods in Computer Science*, 3(4), 2007.

[13] R. J. van Glabbeek. The Linear Time – Branching Time Spectrum II. In E. Best, editor, *Proceedings of CONCUR '93*, number 715 in LNCS, pages 66–81. Springer Verlag, 1993.

[14] R. J. van Glabbeek, B. Luttik, and N. Trcka. Branching bisimilarity with explicit divergence. *Fundam. Inform.*, 93(4):371–392, 2009.

[15] R. J. van Glabbeek and W. P. Weijland. Branching time and abstraction in bisimulation semantics. *Journal of the ACM*, 43(3):555–600, 1996.

[16] D. Q. Goldin, S. A. Smolka, P. C. Attie, and E. L. Sonderegger. Turing machines, transition systems, and interaction. *Inf. Comput.*, 194(2):101–128, 2004.

[17] D. Harel and A. Pnueli. On the development of reactive systems. In K. R. Apt, editor, *Logics and Models of Concurrent Systems*, volume F-13 of *NATO ASI Series*, pages 477–498, New York, 1985. Springer-Verlag.

[18] J. E. Hopcroft, R. Motwani, and J. D. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Pearson, 2006.

[19] I. C. C. Phillips. A note on expressiveness of process algebra. In G. L. Burn, S. Gay, and M. D. Ryan, editors, *Proceedings of the First Imperial College Department of Computing Workshop on Theory and Formal Methods*, Workshops in Computing, pages 260–264. Springer-Verlag, 1993.

[20] G. D. Plotkin. A structural approach to operational semantics. *J. Log. Algebr. Program.*, 60-61:17–139, 2004.

[21] H. Rogers. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill Book Company, 1967. Reprinted by MIT Press, 1987.

[22] M. Sipser. *Introduction to the theory of computation*. PWS Publishing Company, 1997.

[23] A. M. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 2(42):230–265, 1936.

[24] A. M. Turing. Systems of logic based on ordinals. *Proceedings of the London Mathematical Society*, 45(2):161–228, 1939.

[25] F. W. Vaandrager. Expressiveness results for process algebras. In J. W. de Bakker, W. P. de Roever, and G. Rozenberg, editors, *REX Workshop*, volume 666 of *Lecture Notes in Computer Science*, pages 609–638. Springer, 1992.

[26] J. van Leeuwen and J. Wiedermann. On algorithms and interaction. In M. Nielsen and B. Rovan, editors, *MFCS*, volume 1893 of *Lecture Notes in Computer Science*, pages 99–113. Springer, 2000.